Fill in a few small details

Last semester

- An R-module is Noetherian if its submodules satisfy the ACC
- Equivalent condition: All submodules are f.g.
- A ring R is Noeth if it is Noeth as an R-module.
- R Noeth $\Leftrightarrow$ all ideals of R are f.g.

$\boxed{\text{PID} \Rightarrow \text{Noetherian}}$ but UFD $\not\Rightarrow$ Noeth.

Lem R a PID, $\mathfrak{p} \triangleleft R$ prime, $\mathfrak{p} \neq 0 \Rightarrow \mathfrak{p}$ max'l.

Pf: if $\mathfrak{p} = (a)$ is prime ideal, $\neq 0$, choose $\mathfrak{p} \subset m \triangleleft R$

m max'l. $m = (b)$ $a = br$

a prime $\Rightarrow a \mid b$ or $a \mid r$ $r = as \Rightarrow a = bas$

$\Downarrow$ $1 = bs \Rightarrow b \in R^\times$

$(b) \subset (a)$ $\Downarrow$ $(b) = m$ max'l $\lightning$

Back to last time:

Thm Let R be a UFD, then $R[x]$ is a UFD.

Pf: Let f be irred. Want to show f is prime in $R[x]$.

Last time: showed that if $f = r \in R$ irred then

r prime in $R[x]$.

If $f \notin R$ then f irred $\Rightarrow$ f is primitive.

Let $F = \text{frac } R$

Lemma 2 (below) $\Rightarrow$ $f(x)$ irred in $F[x]$.

[**Lemma 2 (Gauss)** R UFD, $F = \text{frac } R$
$f \in R[x]$, then f irred in $R[x] \Rightarrow f$ irred in $F[x]$.]

Consider $$\frac{R[x]}{fR[x]} \xrightarrow{\varphi} \frac{F[x]}{fF[x]}$$

domain
(f irred in $F[x]$ PID
$\Rightarrow f$ prime in $F[x]$)

if $[g] \in \ker \varphi$ $g \in R[x]$
then $g \in fF[x]$ but lem 1 $\Rightarrow$ $g \in fR[x]$
so $[g] = 0 \Rightarrow$ inj. $\square$

Lem 1: $f, g \in R[x]$, $f | g$ in $F[x]$, f primitive $\Rightarrow$
$f | g$ in $R[x]$.

Exercise Sidebar

Let R UFD $f \in R[x]$ primitive means $r | f$ $r \in R \Rightarrow r \in R^*$

Practice: • Show $f \in R[x]$ factors as $f = rg$ w/
$r \in R$, g primitive

• Show if $f \in R[x]$, $F = \text{frac } R$ then $\exists \lambda \in F$
s.t. λf primitive, λ unique up to mult. by units in R.

Pf of Lem 1:

Suppose f primitive, $f, g \in R[x]$, $f | g$ in $F[x]$

write $g = fh$ $h \in F[x]$. $h = \lambda h_0$ h_0 primitive. $\lambda \in F$

$\lambda = \frac{a}{b}$ w/ a, b, having no common irred. factors

then $bg = afh_0 = f(ah_0)$

Claim: $b \in R^\times$

If not, can find π prime $\pi | b \Rightarrow \pi | afh_0$

by first part of proof of thm, know π prime in R

$\Rightarrow \pi$ prime in $R[x]$

$\Rightarrow \pi | afh_0 \Rightarrow \pi | a$ or $\pi | f$ or $\pi | h_0$

no since a & b have no common factors & $\pi | b$.

no, since f is primitive

no since h_0 primitive

$b \in R^\times \Rightarrow bg = afh_0 \Rightarrow g = f\underbrace{(ah_0b^{-1})}_{\in R[x]} = h$ ✓ $\square$.

Lem 2 (Gauss' Lemma)

R UFD, F frac(R) $f \in R[x]$. Then f irred in $R[x]$

$\Rightarrow f$ irred in $F[x]$.

Pf: by contradiction, assume $f = gh$ in $F[x]$

WTS $f = g_0 h_0$ in $R[x]$.

write $g = \lambda g_0$ for g_0 primitive.

$f = g_0 \underbrace{(\lambda h_0)}_{\text{in } R[x]}$ $\quad$ g_0 primitive, $f, g_0 \in R[x]$

$g_0 \mid f$ in $F[x]$ $\overset{\text{Lem 1}}{\Longrightarrow}$ $g_0 \mid f$ in $R[x]$

$\square$.

Honorable mention:

<u>Eisenstein criterion</u>

If R comm. domain, $\pi \in R$ prime element.

then if $f = \sum_{i=0}^{n} a_i x^i$ $\quad$ $a_n \in R^\times$, $\pi \mid a_i$ $i = 0, \dots, n-1$

$\pi^2 \nmid a_0 \Rightarrow f$ irred.

<u>Pf</u>: If $f = gh$, consider images in $R/\pi R[x]$

and in $\underbrace{\mathrm{frac}(R/\pi R)}_{K \text{ field}}[x]$ $\Big\}$ PID. UFD

$\bar{f} = \underset{\substack{\uparrow \\ \text{unit.}}}{a_n} x^n$ $\quad$ x prime in $K[x]$

$\bar{g}\bar{h}$ $\quad$ $\bar{g} = bx^i$ $\quad$ $\bar{h} = cx^j$

$i, j > 0$ (if $r \mid f \Rightarrow r \mid a_n \in R^\times$ r unit.)

there const. coeffs of both g & h are in πR.

$\Rightarrow$ const coeff of $\underset{\substack{\| \\ f}}{gh} \in \pi^2 R$ $\Downarrow$.

Field (Extensions)

f polynomial coeff in ?

Is $f(a) = 0$? $a \in R \hookrightarrow F = \mathrm{frac}\, R$

want R a domain for these answers to be well behaved.

$$K[x_1, \dots, x_n] \xrightarrow{ev_{\vec{a}}} R$$

$(a_1, \dots, a_n) \in R \supseteq K$

$$f \longmapsto f(a)$$

Field extensions

Let $F \subset E$ be a field extension.

if $\alpha \in E$ we say α is algebraic over F if $\exists$ poly $f(x) = F[x]$ s.t. $f(\alpha) = 0$.

Otherwise, we say α is transcendental.

Given $\alpha \in E$

evaluation: $F[x] \xrightarrow{ev_\alpha} E$, $f \longmapsto f(\alpha)$

E domain (field) $\Rightarrow$ $\ker(ev_\alpha)$ is prime (so maxl)

$F[x]$ PID. so gen by some poly

$f_\alpha \in F[x]$.
$\parallel$
$\min_F(\alpha)$

and note:

$$\begin{tikzcd}
F[x] \arrow[r, "ev_\alpha"] \arrow[d] \arrow[dd, bend right] & E \\
F[x]/\min_F\alpha \arrow[ur] \arrow[d, phantom, "\cong"] & \\
F(\alpha) \arrow[uur, bend right] &
\end{tikzcd}$$

$F[x]/\min_F\alpha \cong$ im of ev_α

Observations:

If E/F $\alpha \in E$ is algebraic over F then $\exists!$ monic poly $\min_F\alpha$ s.t. $F(\alpha) \cong \dfrac{F[x]}{\min_F\alpha}$

$F(\alpha)$ = subfield of E gen by F, α

and consequently as $\dim_F \dfrac{F[x]}{\min_F\alpha} = \text{degree}(\min_F\alpha)$

we have $F(\alpha)$ f.d.'l over F.

basis $1, x, x^2, \dots, x^{d-1}$ $\quad d = \text{degree}.\min_F\alpha$

$\min_F\alpha = x^d + a_{d-1}x^{d-1} + \dots + a_0$

$x^d = -a_{d-1}x^{d-1} - \dots - a_0$

$(F[x])^* = F^* \Rightarrow \min_F\alpha$ unique as a monic poly.

In fact:

Lem $\alpha \in E \supset F$ alg. over F iff $\dim_F F(\alpha) < \infty$

Notation: $\dim_F L = [L:F]$

Pf: if $d = [F(\alpha):F]$ then $1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^d$ dependent over F

choose m minil s.t. $1, \alpha, \ldots, \alpha^m$ dependent.

$\Rightarrow \alpha^m = -a_0 - a_1\alpha - a_2\alpha^2 - \cdots - a_{m-1}\alpha^{m-1}$

$\Rightarrow \alpha$ root of $\sum_{i=0}^{m} a_i x^i$.

Notation: If $\alpha \in E \supset F$

$F[\alpha]$ = subring of E gen by F & α

$F(\alpha)$ = subfield of E gen by F & α

Rem: if α is algebraic over F then $F[\alpha] = F(\alpha)$.

Lem: If R is a domain conting F & $\dim_F R < \infty$ then R is a field.

Pf: if $r \in R \setminus \{0\}$ then $\text{mult}_r : R \to R$ is injective (domain) $\Rightarrow$ isom.

$\Rightarrow \text{mult}_r(s) = 1 \Rightarrow rs = 1$ some $s \in R$, $r \in R^\times$ $\square$.