Last time: introduced algebraic closures

- Defined last time: $E \supset F$ is an alg. closure if E is algebraic over F & if every poly $f \in F[x]$ splits in $E[x]$

$\Updownarrow$ HW

$E \supset F$ alg. & every poly $f \in E[x]$ splits in $E[x]$

$\Updownarrow$ HW

$E \supset F$ alg. & every poly $f \in E[x]$ has a root in $E[x]$.

$\Updownarrow$ exercise

E admits no proper algebraic extensions.

Galois theory

Intuitively: fields ⟷ top spaces (manifolds) ("very hyperbolic" Riemannian)

Gal exts ⟷ cov. spaces

K / $\mathbb{Q}$ ⇝ Ab extensions } class field thy.

$\mathrm{Gal}(\mathbb{Q}) \longrightarrow C_n \subset \mathbb{C}^* = GL_1(\mathbb{C})$

$\cup$ S^1 $\quad 1 \to n \quad GL_n(\mathbb{C})$ "Langlands"

$GL_2(\mathbb{C})$

$S^1 \times S^1$?

Galois correspondence

Consider a field ext E/F $\quad Gal(E/F) = \{\text{auts } \varphi : E \to E \mid \varphi|_F = id_F\}$

$$\left\{\begin{matrix}\text{Intermediate field} \\ \text{exts } L\end{matrix}\right\} \underset{\underset{F}{Fix}}{\overset{\overset{G}{Gal}}{\rightleftarrows}} \{\text{subgps } H < Gal(E/F)\}$$

$F \subset L \subset E$

$$H \rightsquigarrow Fix(H) = \{a \in E \mid \sigma(a) = a \text{ all } \sigma \in H\} = E^H$$

$$L \rightsquigarrow Gal(E/L)$$

$$H_1 \subset H_2 \Rightarrow F(H_1) \supset F(H_2)$$

$$L_1 \subset L_2 \Rightarrow G(L_1) \supset G(L_2)$$

$$H \subset G(F(H)) \qquad L \subset F(G(L))$$

Def A Galois correspondence between two PO sets $\mathcal{F}$ $\mathcal{G}$ is a pair of order reversing maps $F : \mathcal{G} \to \mathcal{F}$, $G : \mathcal{F} \to \mathcal{G}$

s.t. $g \leq GF(g)$ & $f \leq FG(f)$ all $g \in \mathcal{G}$, $f \in \mathcal{F}$.

Exercise: if $g \in G$ then $\quad Fg = FGFg$

$$Fg \subset FG(Fg)$$

$$g \subset GFg \Rightarrow Fg \supset FGFg.$$

Cor: $FGFG = FG \qquad GFGF = GF$

Def: if $e: X \to X$ map w/ $e^2 = e$ we say e is a closure operator and if $x \in X$ s.t. $ex = x$ we say x is (e)-closed.

Prop: If $\mathcal{F} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{G}$ is a Gal. correspondence, then $f \in \mathcal{F}$ is closed iff $f = Fg$ some $g \in G$ (sim w/ f, g reversed) & F & G are bijective on closed elements.

ex: $S = \{\text{subsets on } \mathbb{C}^n\}$ $P = \{\text{subsets of } \mathbb{C}[x_1, \dots, x_n]\}$

$$S \longrightarrow P$$

$$\underset{\mathbb{C}^n}{Z} \rightsquigarrow \{f \mid f(z) = 0 \text{ all } z \in Z\}$$

$$\{z \in \mathbb{C}^n \mid f(z) = 0 \text{ all } f \in \mathcal{L}\} \leftarrow \underset{\mathbb{C}[x_1, \dots, x_n]}{\mathcal{L}}$$

closed in S "alg sets" closed in P "radical ideals"

Def E/F is Galois if it is finite & F closed.

(equiv. $F = E^{\mathrm{Gal}(E/F)}$)

Galois groups

If E/F finite extension, $f \in F[x]$, $G = \mathrm{Gal}(E/F)$

then G permutes the roots of f which lie in E.

i.e. if $\alpha \in E$ a root of f, $\sigma \in G$ then $\sigma(\alpha)$ a root of f.

$$f(\alpha) = 0 \qquad \underset{\underset{\sigma(0)=0}{\shortparallel}}{\sigma(f(\alpha))} = f(\sigma(\alpha)) = 0$$

In the special case that E is generated as a field by roots of f, we get $G \hookrightarrow S_{\{\text{roots of } f \text{ in } E\}}$

If f is irreducible$/F$ & E is a splitting field over F of some $g \in F[x]$

then G acts transitively on roots of f lying in E.

i.e. if $\alpha, \beta \in E$ are roots of f then $\exists\, \sigma \in G$ s.t. $\sigma(\alpha) = \beta$.

$$\begin{aligned} E = \text{sp. field of } g \text{ over } F &= \text{sp. field of } g \text{ over } F(\alpha) \\ &= \text{sp. field of } g \text{ over } F(\beta) \end{aligned}$$

$F(\alpha) \cong F(\beta) = F[x]/f \Rightarrow$ sp. fields iso also.

$\exists$ iso $\underset{\text{sp. field}/F(\alpha)}{E} \longrightarrow E = \text{sp. field}/F(\beta)$

$\alpha \longmapsto \beta$

Splitting fields are automorphism stable

Ex: $F \subset K \subset E$ may have F-alg. maps $K \to E$ which don't preserve K.

$F = \mathbb{Q}$ $K = \mathbb{Q}(\sqrt[3]{2})$ $E = \mathbb{C}$

$K' = \mathbb{Q}(\rho \sqrt[3]{2})$

$\mathbb{Q}$-alg map $K \to \mathbb{C}$, $\xrightarrow{\sim} K' \subset \mathbb{C}$

$\rho = e^{2\pi i/3}$

$x^3 - 2$

Def. K/F is Aut stable if for any E/K, any F-alg. hom. $\varphi : K \to E$ takes K to itself.

Lem. If E/F is a splitting field for some poly $f \in F[x]$ Then E is Aut. stable.

Pf: E gen by roots of f, F-homs take roots of f to roots of f. $\square$.

Normal extensions

An algebraic ext. E/F is called normal if whenever $f \in F[x]$ irred has a root in E, f splits in E.

ex: • E = alg. closure of F

• E/F degree 2. $f \in F$ irred w/ root α in E

f deg $2 \Rightarrow (x-\alpha)g = f$

$\min_F(\alpha)$

$\frac{F[x]}{\min_F \alpha} \cong F(\alpha) \subset E$

$x - \alpha$.

Lem: finite normal exts are splitting fields.

Pf: E/F normal, choose $\alpha_1, \dots, \alpha_n \in E$ generating E as a field. Let $f = f_1 \cdots f_n$ $\quad f_i = \min_F \alpha_i$

Know: f_i has a root $\alpha_i \in E$ so splits in E.

$\Rightarrow f$ splits in E. $\quad L$ = splitting field for f $\quad F \subset L \subset E$

but $\alpha_i \in L$, α_i gen $E \Rightarrow L = E$ is a splitting field.

Lem If E/F is f.dim'l. Aut-stable then it is normal.

Pf: WTS if f irred in $F[x]$, $\alpha \in E$ root of f, then f splits in E.

Construct L/F splitting field of $f \cdot g$ $\quad g = \prod g_i \quad g_i = \min_F \alpha_i$

α_i root of g_i

$\alpha_1, \dots, \alpha_n$ lin. gen set for E/F.

$$\begin{tikzcd}
\beta \in L \arrow[rr, "\alpha \mapsto \beta"] & & L \\
\alpha \in E \arrow[u, dash] \arrow[rr] \arrow[rr, bend left] \arrow[dr, dash] & & \beta \in E \arrow[u, dash] \arrow[dl, dash] \\
& F &
\end{tikzcd}$$

L = sp. field$/F(\alpha)$ or $/F(\beta)$

$\square$.