Analogy

Sep. extension $\longleftrightarrow$ finite sets.

$S \rightsquigarrow P(S)$

$\rightsquigarrow \binom{S}{r} = \{\text{subsets } T \subset S,\ |T| = r\}$

$\rightsquigarrow$ various types of partitions of S

Problem: Given $f \in F[x]$ $\deg f = n = s+t$

construct a poly $g_{t,s}$ s.t. $g_{t,s}$ has a root in a field ext. L/F

$\iff$ f factors as $h_t h_s$ $\deg h_t = t$ $\deg h_s = s$ in $L[x]$.

ex: $t=1$ $s=n-1$ $\rightsquigarrow g_{1,n-1} = f$

$n=4$ $t=2, s=2$

$f = x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$

$h_t = x^2 + ax + b$ $\quad h_s = x^2 + cx + d$

$h_s h_t = x^4 + (a+c)x^3 + (d + ac + b)x^2 + (ad + bc)x + bd$

$a + c = \alpha_3$

solve for a,b,c,d $\quad a = \alpha_3 - c$

(HW?)

Overview of classification of field extensions

Transcendental $\qquad\qquad$ Algebraic

$\qquad\qquad\qquad\qquad$ separable $\quad$ inseparable

Def E/F is totally transcendental if $\forall \alpha \in E \setminus F$, α is transcendental over F

Def E/F is purely transcendental if $E \cong F(x_1)(x_2)\cdots(x_n)$
$$= F(x_1, \ldots, x_n)$$

HW: $F(x_1)(x_2)\cdots(x_n) = \left(\operatorname{frac}\left(\operatorname{frac} F[x_1]\right)[x_2]\right) \cdots$

variables (transcendental)

$$= \operatorname{frac}\left(F[x_1, \ldots, x_n]\right)$$

$F(x)/F$

is pt.-transcendental

$$\frac{(x-1)^2}{x+1} = y$$

Claim: $F(x)/F(y)$ finite algebraic.

x a root of $(T-1)^2 - y(T+1)$
$$\in F(y)[T]$$

$\Rightarrow$ y transcendental$/F$

else $F(x)/F(y)$ finite, $F(y)/F$ finite $\Rightarrow$

$F(x)/F$ finite $\times$

$\exists$ E/F tot. transcendental, f.generated but not purely trans. ?

yes, but very subtle.

$\mathbb{C}/\mathbb{Z}^2$

$$\operatorname{frac}\left(F[x,y] \big/ y^2 - x^3 - 1 \right)$$

$z \in$

$\cup$

$F(z)$ f.g. field ext. w/ 1 trans. elt.

E

$|$ 1 trk.

$F(t)$

$|$

F $\quad F = \mathbb{C}$

<u>Luroth problem</u>

if E/F purely transcendental, and $F \subset L \subset E$

is L/F purely transcendental?

<u>Yes</u>: if $E = F(t)$ $\leftarrow$ we'll prob. see

<u>Yes</u>: if $E = \mathbb{C}(s,t) / F = \mathbb{C}$

No! 1970s. $E = \mathbb{C}(s,t,u)$ <u>fails</u>.

Griffiths-Harris "Intermediate Jacobians"

Iskovskikh-Manin. "rigidity" finite aut. gps.

Artin-Mumford "Brauer gp" of elliptic fibred three-folds.

Tensor digression

Properties of $\otimes$.

Given R-algebras A, B can form $A \otimes_R B$

algebra structure given by:

$$\Big(\sum_i a_i \otimes b_i\Big)\Big(\sum_j c_j \otimes d_j\Big) = \sum_{ij} a_i c_j \otimes b_i d_j$$

get $A, B \longrightarrow A \otimes_R B$ & if A & B are commutative

then it has universal property:

if C is any R-algebra w/ R-alg homs $A, B \longrightarrow C$

then $\exists!\ A \otimes_R B \to C$ s.t. diagram commutes

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left] & & \\
& A \otimes B \arrow[r] & C \\
B \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

} coproduct perspective

note $A \to C$ R-alg. map means

$$\begin{tikzcd}
& A \arrow[dr] & \\
R \arrow[ur] \arrow[rr] & & C
\end{tikzcd}$$

commutes.

alt. have

$$\begin{tikzcd}
& A \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & A \otimes B \\
& B \arrow[ur] &
\end{tikzcd}$$

} pushout perspective

s.t. if

$$\begin{tikzcd}
& A \arrow[dr] & \\
R \arrow[ur] \arrow[dr] & & C \\
& B \arrow[ur] &
\end{tikzcd}$$

then have $A \otimes B \to C$

s.t.

$$\begin{tikzcd} & A \arrow[dr] \arrow[drr, bend left] & & \\ R \arrow[ur] \arrow[dr] & & A\otimes B \arrow[r] & C \\ & B \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd}$$

commutes

<u>Recall:</u>

$$R\otimes_R R \cong R \qquad M\otimes_R (N_1\oplus N_2) \cong (M\otimes_R N_1)\oplus(M\otimes_R N_2)$$

in particular $\underset{e_i}{R^n}\otimes_R \underset{f_j}{R^m} \cong R^{nm}$ $\qquad e_if_j = e_i\otimes f_j$

<u>Ex:</u> $$R[x_1,\dots,x_n]/I \otimes_R R[y_1,\dots,y_m]/J \cong \frac{R[x_1,\dots,x_n,y_1,\dots,y_m]}{I\cdot R[\vec{x},\vec{y}] + J R[\vec{x},\vec{y}]}$$

<u>ex:</u> $\mathbb{Q}(\sqrt{2})\otimes_{\mathbb{Q}}\mathbb{R}$ $\qquad \mathbb{Q}\otimes_{\mathbb{Q}}\mathbb{R}\cong\mathbb{R}$

$\cong$

$$\frac{\mathbb{Q}[x]}{x^2-2}\otimes_{\mathbb{Q}}\mathbb{R} \underset{\substack{\uparrow \\ \text{HW}}}{\cong} \frac{\mathbb{R}[x]}{x^2-2}$$

$$\cong \frac{\mathbb{R}[x]}{(x-\sqrt{2})(x+\sqrt{2})}$$

$$\cong \frac{\mathbb{R}[x]}{(x-\sqrt{2})}\times\frac{\mathbb{R}[x]}{x+\sqrt{2}} \cong \mathbb{R}\times\mathbb{R}.$$

$$\frac{R[x_1,\dots,x_n]}{I}\otimes_R S \leftarrow R\xrightarrow{\varphi} S$$

$\cong$

$$\frac{S[x_1,\dots,x_n]}{\varphi(I)}$$

Def if
$$\begin{tikzcd} & L \arrow[dl, dash] \arrow[dr, dash] & \\ E \arrow[dr, dash] & & K \arrow[dl, dash] \\ & F & \end{tikzcd}$$
field exts. define $\langle E,K\rangle = \langle E,K\rangle_L$ is the field generated by E & K "compositum" (in L)

diagram of fields $\subset \supset$ / $\supset \subset$ upwards inclusions

Note: $E, K \longrightarrow \langle E,K\rangle$ F-alg maps.

$\Rightarrow \exists$ map $E \otimes_F K \xrightarrow{\varphi} \langle E,K\rangle$

Def E & K are linearly disjoint if φ is injective.

Image of φ = sub ring generated by E & K.

if $\langle E,K\rangle / F$ finite dim'l. $\Rightarrow$ im φ f.dim'l $/F$ $\Rightarrow$ its a field.

Rem: if $E, K/F$ finite then injective $\Rightarrow$ isomorphism.

Separable & inseparable/perfect closures

If E / F algebraic then $E^{sep} = \{\alpha \in E \mid \min_F \alpha \text{ separable}\}$

char $F = p$

$E^{insep} = \{\alpha \in E \mid \alpha^{p^n} \in F \text{ some } n\}$

then E^{sep} E^{insep} are subfields.

if $\alpha, \beta \in$ (?)

wts $F(\alpha,\beta) \in E^{sep}$ ~~$f = \min_F \alpha$ $g = \min_F \beta$~~

~~$K = \text{Split}(fg)$~~ ~~K / F separable.~~

~~and $\alpha, \beta \in K$~~

$F(\alpha)$ $\quad g = \min_F \beta \quad \bar{g} = \min_{F(\alpha)} \beta \mid g$ sep

$F(\alpha)(\beta)/F(\alpha)$ separable.

and separable/separable = separable.

$\alpha^{p^n}, \beta^{p^m} \in F \qquad (\alpha+\beta)^{p^{n+m}} = \alpha^{p^{n+m}} + \beta^{p^{n+m}} \cdots \in F$

$$\begin{tikzcd}
 & E \arrow[dl, dash] \arrow[dr, dash] & \\
E^{insep} \arrow[dr, dash] & & E^{sep} \arrow[dl, dash] \\
 & F &
\end{tikzcd}$$