

Analogy

Sep. extensions \longleftrightarrow finite sets.

$$S \rightsquigarrow \mathcal{P}(S)$$

$$\rightsquigarrow \binom{S}{r} = \left\{ \text{subsets } T \subset S \right. \\ \left. |T| = r \right\}$$

\rightsquigarrow various types of partitions of S

Problem: Given $f \in F[x]$ $\deg f = n = s+t$

construct a poly $g_{t,s}$ s.t. $g_{t,s}$ has a root in a field ext. L/F

$\Leftrightarrow f$ factors as $h_t h_s$ $\deg h_t = t$ $\deg h_s = s$ in $L[x]$.

ex: $t=1$ $s=n-1 \rightsquigarrow g_{1,n-1} = f$

$$n=4 \quad t=2, s=2$$

$$f = x^4 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0$$

$$h_t = x^2 + ax + b \quad h_s = x^2 + cx + d$$

$$h_s h_t = x^4 + (a+c)x^3 + (d+ac+b)x^2 + (ad+bc)x + bd$$

solve for a, b, c, d

$$a+c = \alpha_3$$

$$a = \alpha_3 - c$$

(HW?)

Overview of classification of field extensions

Transcendental

Algebraic

separable inseparable

Def E/F is totally transcendental if $\forall \alpha \in E \setminus F$
 α is transcendental over F

Def E/F is purely transcendental if $E \cong F(x_1)(x_2)\dots(x_n)$
 $= F(x_1, \dots, x_n)$

HW: $F(x_1)(x_2)\dots(x_n) = \text{frac}(\text{frac}(F[x_1])\dots[x_2]) \dots$
 \uparrow
 variables (transcendental)
 $= \text{frac}(F[x_1, \dots, x_n])$

$F(x)/F$

$$\frac{(x-1)^2}{x+1} = y$$

\uparrow
is pt-transcendental

Claim: $F(x)/F(y)$ finite algebraic.

x a root of $(T-1)^2 - y(T+1)$

$$\in F(y)[T]$$

$\Rightarrow y$ transcendental/ F

else $F(x)/F(y)$ finite $F(y)/F$ finite \Rightarrow

$F(x)/F$ finite \times

$\exists E/F$ tot. transcendental, \mathbb{A}^1 -generated but not purely trans. ?
 yes, but very subtle.

$\text{Spec} \left(\mathbb{F}[x,y] / (y^2 - x^3 - 1) \right) \xrightarrow{\sim} \mathbb{C}/\mathbb{Z}^2$

$\cong \mathbb{F}(z)$ f.s. isodent. w/ 1 transc. elem.

$\begin{array}{c} \mathbb{F} \\ \text{fht.} \\ \mathbb{F}(t) \\ \downarrow \\ \mathbb{F} \end{array} \quad \mathbb{F} = \mathbb{C}$

Luroth problem

if E/F purely transcendental, and $\mathbb{F} \subset L \subset E$
 is L/F purely transcendental?

Yes: if $E = \mathbb{F}(t)$ \leftarrow well prob. case

Yes: if $E = \mathbb{C}(s,t) / \mathbb{F} = \mathbb{C}$

No! 1970s. $E = \mathbb{C}(s,t,u)$ fails.

Griffiths-Harris
 "Infinite
 Jacobians"

Isvaich-Mann.
 "rigidity"
 Arith. aut. gps.

Artin-Mumford
 "Brauer sp" of elliptic fibred
 free fib.

Tensor Algebra

Properties of \otimes

Given R -algebras A, B can form $A \otimes_R B$
 algebra structure given by:

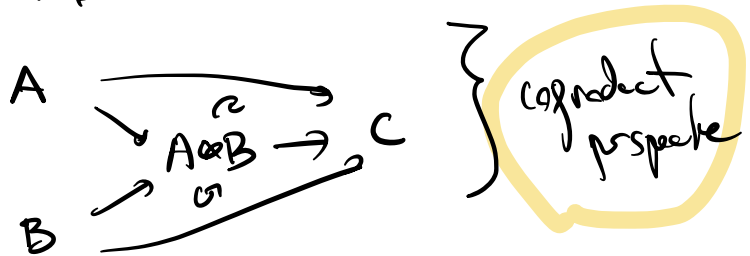
$$\left(\sum_i a_i \otimes b_i\right) \left(\sum_j c_j \otimes d_j\right) = \sum_{ij} a_i c_j \otimes b_i d_j$$

get $A, B \rightarrow A \otimes_R B$ if A, B are commutative

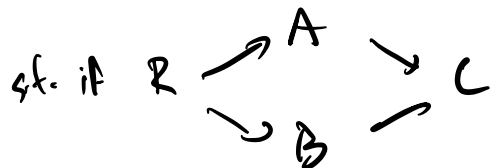
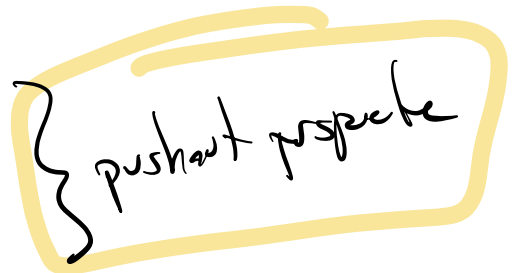
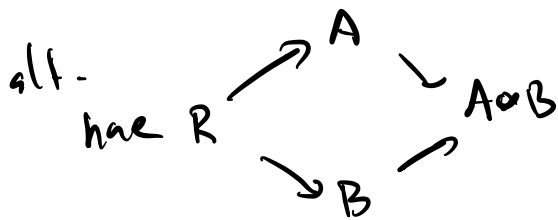
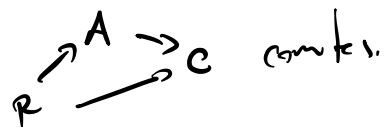
then it has universal property:

if C is any R -algebra w/ homs $A, B \rightarrow C$

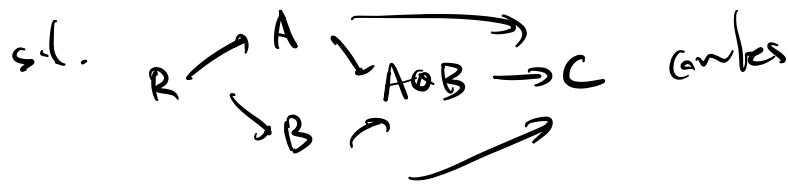
then $\exists!$ $A \otimes_R B \rightarrow C$ s.t. diagram commutes



note $A \rightarrow C$ R -alg. map means



then have $A \otimes B \rightarrow C$



Recall:

$$R \otimes_R R \cong R$$

$$M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$$

in particular $R^n \otimes_R R^m \cong R^{nm}$

$$e_i \otimes f_j = e_i \otimes f_j$$

Ex: $R[x_1, \dots, x_n] \otimes_R R[y_1, \dots, y_m] \cong R[x_1, \dots, x_n, y_1, \dots, y_m]$

ex: $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{R}$

$$\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}$$

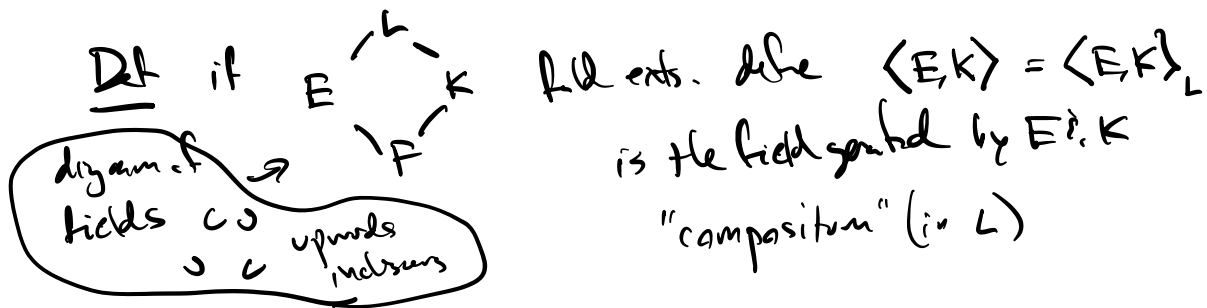
$$\frac{\mathbb{Q}[x]}{x^2-2} \otimes_{\mathbb{Q}} \mathbb{R} \cong \frac{\mathbb{R}[x]}{x^2-2}$$

HW

$$\cong \frac{\mathbb{R}[x]}{(x-\sqrt{2})(x+\sqrt{2})}$$

$$\cong \frac{\mathbb{R}[x]}{x-\sqrt{2}} \times \frac{\mathbb{R}[x]}{x+\sqrt{2}} \cong \mathbb{R} \times \mathbb{R}$$

$$\begin{array}{ccc}
 \mathbb{R}[x_1, \dots, x_n] \otimes_{\mathbb{R}} \mathbb{S} & \xleftarrow{\quad} & \mathbb{R} \xrightarrow{\quad} \mathbb{S} \\
 \downarrow & & \downarrow \\
 \mathbb{S}[x_1, \dots, x_n] & & \mathbb{S}
 \end{array}$$



Note: $E, K \rightarrow \langle E, K \rangle$ F -alg maps.

$\Rightarrow \exists$ map $E \otimes_F K \xrightarrow{\varphi} \langle E, K \rangle$

Def E, K are linearly disjoint if φ is injective.

Image of $\varphi =$ subring generated by E, K .

if $\langle E, K \rangle / F$ finite dim. \Rightarrow im of φ finite dim. \Rightarrow its a field.

Prop: if $E, K / F$ finite then injective \Rightarrow isomorphism.

Separable & inseparable / perfect closures

if E / F algebraic then $E^{sep} = \{ \alpha \in E \mid \text{min}_F \alpha \text{ separable} \}$
 $\text{chr } F = p$ $E^{insep} = \{ \alpha \in E \mid \alpha^{p^n} \in F \text{ some } n \}$

then E^{sep}, E^{insep} are subfields.

if α, β

with $F(\alpha, \beta) \in E^{sep}$

~~$f = \text{min}_F \alpha, g = \text{min}_F \beta$
 f, g separable~~

and $\alpha, \beta \in K$

~~K / F separable.~~

$$F(\alpha) \quad g = \min_{F(\beta)} \quad \bar{g} = \min_{F(\alpha)\beta} \quad |g|$$

up

$F(\alpha)(\beta)/F(\alpha)$ separable.

and separable/separable = separable.

$$\alpha^p, \beta^p \in F \quad (\alpha + \beta)^{p+n} = \alpha^{p+n} + \beta^{p+n} \dots \in F$$

