

Isaacs: Ch 18, Ch 19 recently Later today \rightarrow ch 22

Jacobson's Basic Alg I, II
↓
Basic Gal theory \rightarrow Ch 8

Also see Keith Conrad
--- /blurbhs/

From last time

E/F algebraic extension $\text{char } F = p$.

$$F^{E, s} = E^{sep} = \{ \alpha \in E \mid \text{min}_F \alpha \text{ is separable} \}$$

$$F^{E, i} = E^{imp} = \{ \alpha \in E \mid \alpha^{p^n} \in F \text{ some } n \}$$

Observed last time: E^{imp}, E^{sep} are subfields of E .

Lemma: E/E^{sep} is a purely inseparable extension.

Pf: if $\alpha \in E$ let $f = \text{min}_F \alpha$, $f(x) = g(x^{p^n})$

some n , g irred separable. let $\beta = \alpha^{p^n}$ $g(\beta) = 0$

$g = \text{min}_F \beta \Rightarrow \beta \in E^{sep}$ i.e. $\forall \alpha \in E \alpha^{p^n} \in E^{sep}$ some n .

Def K/F purely inseparable if $\forall \alpha \in K \alpha^{p^n} \in F$ some n

Lemma: E/F normal algebraic extension then E/E^{imp} is a separable extension.

P1: let $\alpha \in E$ $f = \min_P a \Rightarrow g(x^{p^n}) = f(x)$ g sep. irr.

$\beta = \alpha^{p^n}$ $g(\beta) = 0 \Rightarrow g$ has a root in E , E normal

$\Rightarrow g = \prod (x - \beta_i) \Rightarrow f = \prod (x^{p^n} - \beta_i)$

and f has a root in E (α) so f factors into linear factors.

$$(x^{p^n} - \beta) = (x^{p^n} - \alpha^{p^n}) = (x - \alpha)^{p^n}$$

$$\beta_i = \beta$$

$(x^{p^n} - \beta_i)$ factors $\Rightarrow \beta_i = \alpha_i^{p^n}$

$$(x^{p^n} - \beta_i) = (x - \alpha_i)^{p^n}$$

Exercise

hint: has repeated roots since $f' = 0$.

\Rightarrow in E , $f = \prod (x - \alpha_i)^{p^n}$

$$f = (h)^{p^n}$$

$$h = \prod (x - \alpha_i)$$

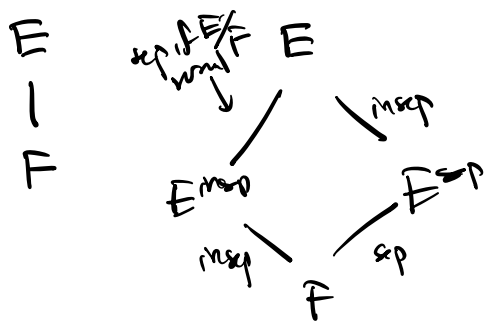
$$= \sum c_i x^i$$

$$f = \sum c_i^{p^n} x^{ip^n} \in F[x]$$

$c_i \in E^{insep} \Rightarrow h \in E^{insep}[x]$

$$h(\alpha) = 0$$

h is separable / E^{insep} \square



(if it exists)

Def given $K \supset F \supset L$ field exts. $\langle K, L \rangle_F$
 (free) composition universal field
 containing both K & L .

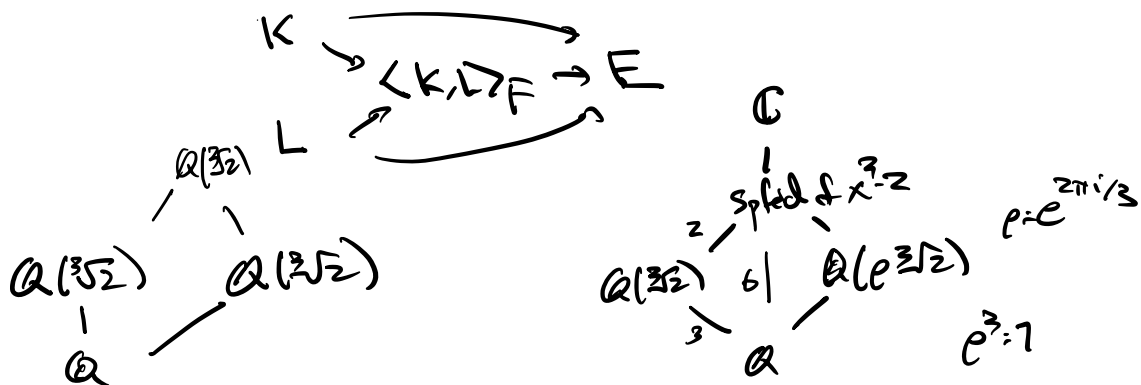
$K \otimes_F L$ "free F -algebra composition"
comm.

Note: if $K \otimes_F L$ is a field then $\langle K, L \rangle_F \cong K \otimes_F L$

Univ. prop. of $\langle K, L \rangle_F$ is: its a field w/ inclusions of F -exts.
 $K, L \rightarrow \langle K, L \rangle_F$

s.t. if E is a field w/ inclusions

$K, L \rightarrow E$ then $\exists!$ inclusion $\langle K, L \rangle_F \rightarrow E$ s.t.



$$\langle \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{2}) \rangle_{\mathbb{Q}} \xrightarrow{\text{sp. fld}} \mathbb{Q}(\sqrt{2}) \quad x^2-2$$

$$\begin{array}{ccc} \mathbb{Q}(\sqrt{2}) & \xrightarrow{\quad} & \sqrt{2} \\ \downarrow \mathbb{Q}(\sqrt{2}) & \xrightarrow{\quad} & \mathbb{Q} \\ \mathbb{Q}(\sqrt{2}) & \xrightarrow{\quad} & \mathbb{Q}(\sqrt{2}) \end{array}$$

$$\mathbb{Q}(\sqrt{2}) \rightarrow \langle \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{2}) \rangle_{\mathbb{Q}} \xrightarrow{\text{id}} \mathbb{Q}(\sqrt{2})$$

" $\mathbb{Q}(\sqrt{2})$ if exists

HW: If K/F sep. alg. & L/F is inseparable alg. ext. then $K \otimes_F L$ is a domain (actually a field)

Def $\langle K, L \rangle_F^E = KL$
 = subfld of E
 generated by K & L .

$$\begin{array}{ccc} & E & \\ & / \quad \backslash & \\ K & & L \\ & \backslash \quad / & \\ & F & \end{array}$$

Note: \exists canonical map $\langle K, L \rangle_F \rightarrow \langle K, L \rangle_F^{\overline{F}}$
Q: when is this an \mathbb{Q} ?

Recall: if $K \not\subseteq L$ \swarrow \nearrow
 injecte we say K, L are
 linearly disjoint.

HW: $\langle K, L \rangle_F$ exists iff $K \not\subseteq L$ has a unique maximal ideal.
 for K, L subfields of F

Theorem: (Isaacs 18.22)

Suppose we have field extensions w/ $LE = K$ $LN = M$
 and E/M is ^{triv.} Galois. Then



K/L is Galois & the map

$$\text{Gal}(K/L) \rightarrow \text{Gal}(E/M)$$

$$\sigma \mapsto \sigma|_E \text{ is an iso.}$$

and L & E are linearly disjoint over M .

Next stage in our journey?

Galois Descent \rightarrow Galois Cohom \rightarrow Other properties on
 Galois extensions
 "extension problem"

Target result:

Def: A field F is called pseudo algebraically closed (PAC)
 if every homogeneous polynomial $f \in F[x_1, \dots, x_n]$ of degree d
 has a nontrivial root when $n > d$.

i.e. $\exists (a_1, \dots, a_n) \in F^n \setminus \{0\}$ s.t. $f(\vec{a}) = 0$. (if $n > d$)

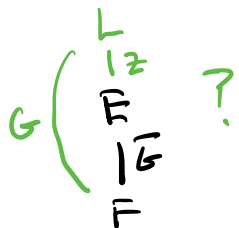
Tseu's thm $\mathbb{C}(t)$ is PAC. \mathbb{F}_q PAC.

Central Extension problem:
embed

If we have a gp G w/ $Z \leq Z(G)$ & $\bar{G} = G/Z$.

and if $\frac{E}{F}$ is Galois w/ gp \bar{G} , can we find a

extension L/E s.t. $\frac{L}{F}$ is G -Galois



Thm if F is PAC then yes.

Strategy: to solve \uparrow embed problem translate to show surject

in $H^2(\bar{G}, Z)$ is 0.

$H^2 \mapsto$ translate to mult. table for an associative algebra.

$\left. \begin{array}{l} \text{Gal. descent of} \\ \text{coborn} \end{array} \right\} \rightarrow$
 $\left. \begin{array}{l} \text{W. Artin} \rightarrow \text{div. alg.} \end{array} \right\} \rightarrow$

$$\text{div. alg.} \neq F \iff H^2 = 0$$

Tseu's thm to construct d divisors on any nonzero div. alg.

For warm up:

- check out my youtube videos

Review:

- Montre theorem
- Group Cohom.
- Wedderburn Artin.