Isaacs: Ch18, Ch19 recently Later today ⇝ ch22

Jacobson's Basic Alg I, II ↙ Basic Gal thy, ↘ Ch8

Also see Keith Conrad ---/blurbs/

From last time:

E/F algebraic extension char $F = p$.

$F^{E,s} = E^{sep} = \{\alpha \in E \mid \min_F \alpha \text{ is separable}\}$

$F^{E,i} = E^{insep} = \{\alpha \in E \mid \alpha^{p^n} \in F \text{ some } n\}$

Observed last time: E^{insep}, E^{sep} are subfields of E.

Lemma: E/E^{sep} is a purely inseparable extension.

Pf: if $\alpha \in E$ let $f = \min_F \alpha$, $f(x) = g(x^{p^n})$ some n, g irred separable. let $\beta = \alpha^{p^n}$ $g(\beta) = 0$

$g = \min_F \beta \Rightarrow \beta \in E^{sep}$ i.e. $\forall \alpha \in E$ $\alpha^{p^n} \in E^{sep}$ some n.

Def L/F purely inseparable if $\forall \alpha \in L$ $\alpha^{p^n} \in F$ some n

Lemma: E/F normal algebraic extension then E/E^{insep} is a separable extension.

Pf: let $\alpha \in E$ $f = \min_F \alpha$ $\Rightarrow$ $g(x^{p^n}) = f(x)$ g sep. irred.

$\beta = \alpha^{p^n}$ $g(\beta) = 0$ $\Rightarrow$ g has a root in E, E normal

$\Rightarrow g = \prod (x - \beta_i) \Rightarrow f = \prod (x^{p^n} - \beta_i)$

and f has a root in E (α) so f facts into lin facts.

$$(x^{p^n} - \beta) = (x^{p^n} - \alpha^{p^n}) = (x - \alpha)^{p^n}$$

$\beta_1 = \beta$ $\quad (x^{p^n} - \beta_i)$ facts $\Rightarrow \beta_i = \alpha_i^{p^n}$

$$(x^{p^n} - \beta_i) = (x - \alpha_i)^{p^n}$$

Exercise

hints: have repeated roots since $f' = 0$.

$\Rightarrow$ in E, $f = \prod (x - \alpha_i)^{p^n}$ $\quad h = \prod (x - \alpha_i)$

$f = (h)^{p^n}$ $\quad = \sum c_i x^i$

$$f = \sum c_i^{p^n} x^{ip^n} \in F[x]$$

$c_i \in E^{insep} \Rightarrow h \in E^{insep}[x]$ $\quad h(\alpha) = 0$

h is separable $/ E^{insep}$ $\square$

$$\begin{array}{c} E \\ | \\ F \end{array}$$

$$\begin{tikzcd}[column sep=small]
 & E \arrow[dl, dash, "\text{sep if } E/F \text{ norml}"'] \arrow[dr, dash, "\text{insep}"] & \\
E^{\text{insep}} & & E^{\text{sep}} \\
 & F \arrow[ul, dash, "\text{insep}"] \arrow[ur, dash, "\text{sep}"'] &
\end{tikzcd}$$

<u>Def</u> given
$$\begin{tikzcd}[column sep=small]
K & & L \\
 & F \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$
field exts. $\langle K, L \rangle_F$ (if it exists)

(free) compositum universal field containing both K & L.

$K \otimes_F L$ "free comm. F-algebra compositum"

<u>Note</u>: if $K \otimes_F L$ is a field then $\langle K, L \rangle_F \simeq K \otimes_F L$

Univ. prop of $\langle K, L \rangle_F$ is: its a field w/ inclusions of F-exts $K, L \to \langle K, L \rangle_F$ s.t. if E is a field w/ inclusions $K, L \to E$ then $\exists!$ inclusion $\langle K, L \rangle_F \to E$ s.t.

$$\begin{tikzcd}
K \arrow[dr] \arrow[drr, bend left] & & \\
 & \langle K, L \rangle_F \arrow[r] & E \\
L \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt[3]{2}) & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[ur, dash] & & \mathbb{Q}(\sqrt[3]{2}) \arrow[ul, dash] \\
\mathbb{Q} \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C} & \\
 & \text{Spfeld of } x^3-2 \arrow[u, dash] & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[ur, dash, "2"] & & \mathbb{Q}(\rho\sqrt[3]{2}) \arrow[ul, dash] \\
 & \mathbb{Q} \arrow[ul, dash, "3"] \arrow[uu, dash, "6"] \arrow[ur, dash] &
\end{tikzcd}$$

$\rho = e^{2\pi i/3}$

$\rho^3 = 1$

$$\langle \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}) \rangle_{\mathbb{Q}} \longrightarrow \mathbb{Q}(\sqrt[3]{2}) \qquad x^3-2$$

sp. field $\nearrow$

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[3]{2}) \arrow[rr, "\sqrt[3]{2}"] \arrow[dr] & & \mathbb{C} \\
& \mathbb{Q}(\sqrt[3]{2}) \arrow[ur] & \\
\mathbb{Q}(\sqrt[3]{2}) \arrow[ur] \arrow[uurr, "\rho\sqrt[3]{2}"'] & &
\end{tikzcd}$$

$\mathbb{Q}(\sqrt[3]{2})$ if exists

$$\begin{tikzcd}
\mathbb{Q}(\sqrt[3]{2}) \arrow[r] \arrow[rr, bend right, "id"'] & \langle \mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}) \rangle_{\mathbb{Q}} \arrow[r] & \mathbb{Q}(\sqrt[3]{2})
\end{tikzcd}$$

<u>HW</u>: If K/F sep. alg. & L/F is any alg. ext. then $K \otimes_F L$ is a domain (actually a field)

$$\begin{tikzcd}
& E & \\
K \arrow[ur, dash] & & L \arrow[ul, dash] \\
& F \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

<u>Def</u> $\langle K, L \rangle_F^E = KL$

= subfield of E generated by K & L.

<u>Note</u>: $\exists$ canonical map $\langle K, L \rangle_F \longrightarrow \langle K, L \rangle_F^E$

<u>Q</u>: when is this an $\cong$?

Recall: if $K \otimes_F L$

injective we say K & L are linearly disjoint.

HW: $\langle K, L \rangle_F$ exists iff $K \otimes_F L$ has a unique maximal ideal.

for K, L finite exts of F

Theorem: (Isaacs 18.22)

Suppose we have field extensions w/ $LE = K$ $L \cap E = M$

$$\begin{tikzcd} & K \arrow[dl, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & & E \arrow[dl, dash] \\ & M & \end{tikzcd}$$

and E/M is finite Galois. Then K/L is Galois & the map

$$\mathrm{Gal}(K/L) \longrightarrow \mathrm{Gal}(E/M)$$
$$\sigma \longmapsto \sigma|_E \text{ is an } \cong.$$

and L & E are linearly disjoint over M.

Next stage in our journey:

Galois Descent $\rightsquigarrow$ Galois cohom $\rightsquigarrow$ Other properties on Galois extensions

"extension problem"

Target result:

Def: A field F is called pseudo-algebraically-closed (PAC) if every homogeneous polynomial $f \in F[x_1, \ldots, x_n]$ of degree d has a nontrivial root when $n > d$.

i.e. $\exists (a_1,\dots,a_n) \in F^n \setminus \{\vec{0}\}$ s.t. $f(\vec{a}) = 0$. (if $n>d$)

Tsen's thm: $\mathbb{C}(t)$ is PAC. $\mathbb{F}_q$ PAC.

Central ~~Extension~~ embedding problem:

If we have a gp G w/ $Z \le Z(G)$ & $\bar{G} = G/Z$.

and if $\begin{array}{c} E \\ | \\ F \end{array}$ is Galois w/ gp $\bar{G}$, can we find a

extension L/E s.t. $\begin{array}{c} L \\ | \\ F \end{array}$ is G-Galois

$$G\left(\begin{array}{c} L \\ |\,Z \\ E \\ |\,\bar{G} \\ F \end{array}\right. \quad ?$$

Thm: if F is PAC then yes.

Strategy: to solve $\nearrow$ embedding problem translate to show an element in $H^2(\bar{G}, Z)$ is 0.

Gal. descent of cohom

$H^2 \rightsquigarrow$ translate to mult. table for an associate algebra. W. Artin $\rightsquigarrow$ div alg.

div. alg $\neq F \iff H^2 = 0$

Tsen's thm to construct 0 divisors in any nontriv div. alg.

For warm up:

- check out my youtube videos

Review:

- Morita theorems
- Group Cohom.
- Wedderburn Artin.