

Danny Krashen

dkrashen.org/algebra1

HW due each Monday

Assumed background: groups, rings, fields,
some modules, various linear alg stuff.

W 3:20-4:40 (Review) HILL 425

Algebra - sets & operations

Binary operation - typical axioms

- associativity
- commutativity
- existence of units
- cancellation / inverses.

Ex: Magma = Set w/ binary operation (M, \cdot)

→ Monoid = Set w/ bin. op, associative, unit

Loop = Set w/ bin op, unit, inverses.

→ Group = loop w/ associativity
 Ab. group = group w/ commutativity.

(\mathbb{Z}, \cdot)

Often, multiple operations

(mgs, etc)

Def An n -ary operation on a set S is

a map $S^n \rightarrow S$

$\underbrace{S \times \dots \times S}_{n \text{ times}}$

0-ary

$\{\emptyset\} \rightarrow S$

Monoid

Set M ,

$\text{id}_M: M \rightarrow M$

$a \mapsto a$

0-ary operation: $1: \{\emptyset\} \rightarrow M$

2-ary op... ; $m: M \times M \rightarrow M$

s.t. $M \times M \times M \rightarrow M$

$m(\text{id}_M \times m) = m(m \times \text{id}_M)$

$$\text{assoc} \rightarrow m(x, m(y, z)) = m(m(x, y), z) \quad \forall x, y, z \in M$$

$$x(yz) = (xy)z \quad xy \equiv m(x, y)$$

$$\text{unit} \rightarrow m(\text{id}_M \times 1) = m(1 \times \text{id}_M) = \text{id}_M$$

$$\forall x \in M \quad m(x, 1(\emptyset)) = m(1(\emptyset), x) = x$$

$$1 \equiv 1(\emptyset) \quad x1 = 1x = x$$

$$xy \equiv m(x, y)$$

Notational Aside

given a product $A \times B$ to define a map

$$C \xrightarrow{f} A \times B$$

$$f(c) = (a, b) \quad a = "f_1(c)"$$

$$b = "f_2(c)"$$

we write $f = f_1 \times f_2$

Similarly define groups

$$0\text{-ary op } e: \{\emptyset\} \rightarrow G \quad e \equiv e(\emptyset)$$

$$1\text{-ary op } \iota: G \rightarrow G \quad g^{-1} \equiv \iota(g)$$

$$2\text{-ary op } m: G \times G \rightarrow G \quad gh \equiv m(g, h)$$

same axioms.

Rings $(R, 1, 0, \cdot, +, (-))$

0-ary 2-ary 1-ary
 ↓ ↓ ↓
 ↓ ↓ ↓

Ω -algebra:

Ω a set of symbols w/ "arities"

$$\Omega: \{m, 2, 1\} \rightarrow \mathbb{Z}_{\geq 0}$$

$$m \mapsto 2$$

$$2 \mapsto 1$$

$$1 \mapsto 0$$

Def an Ω -algebra is a set S w/
maps $\lambda: S^n \rightarrow S$

for each $\lambda \in \Omega$ of arity n .

Def Homomorphisms of Ω alg's.

are fns $S \rightarrow T$ s.t. $\forall \lambda \in \Omega$

$$f(\lambda(s_1, \dots, s_n)) = \lambda(f(s_1), \dots, f(s_n))$$

ex:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f} & \mathbb{R} \times \mathbb{R} \\ & \searrow & \downarrow \text{order} \\ & & \mathbb{0} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{0} \end{array}$$
$$x \longmapsto (0, x)$$

$$x+y \longmapsto (0, x+y) = (0, x) + (0, y)$$

$$xy \longmapsto (0, xy) = (0, x)(0, y)$$

$$1 \longmapsto (0, 1) \neq 1$$

$$d=1 \quad f(\lambda(\emptyset)) = \lambda(\emptyset)$$

$$f(1) = 1$$

Def (Imprecise) A Variety = the collection of Ω algebras for a given Ω , satisfying a set of identities.

Ex: Ω as above $m, 2, 1$
 w/ identities $(xy)z = x(yz)$
 $xx^{-1} = x^{-1}x = 1$
 $1x = x1 = x$

Variety defined by these is called "groups"

Fun: Def A congruence on an Ω -algebra A
 is an Ω -subalgebra of $A \times A$

• Consider G a group $H \triangleleft G$

$$C = \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in H\}$$

Show C is a congruence $\Leftrightarrow H \triangleleft G$

• Consider a ring R , $I \triangleleft (R, +)$

$$C = \{(r_1, r_2) \in R \times R \mid r_1 - r_2 \in I\}$$

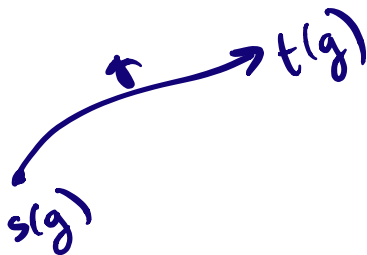
congruence $\Leftrightarrow I \triangleleft R$.

In general, if $A \xrightarrow{f} B$ a hom. of \mathcal{R} -algs
can define ker $f = \{ (a_1, a_2) \mid f(a_1) = f(a_2) \}$
congruences are kernels.

Doesn't capture all types of structures we
care about

Groupoid:

$$G_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} G_0$$



ex: Pick a collection of sets S

$G_1 =$ bijective maps between these sets.