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HW due each Monday

Assumed background: groups, rings, fields, some modules, various linear alg stuff.

W 3:20-4:40 (Review) HILL 425

Algebra – sets & operations

Binary operation ~ typical axioms

- associativity
- commutativity
- existence of units
- cancellation / inverses.

<u>Ex:</u> Magma = Set w/ binary operation $(M, \cdot)$

→ Monoid = Set w/ bin. op, associative, unit

Loop: Set w/ bin op, unit, inverses.

$\to$ Group = loop w/ associativity

Ab. group = group w/ commutativity.

$(2\mathbb{Z}, \cdot)$

Often, multiple operations

(rings, etc)

Def An n-ary operation on a set S is

a map $S^n \longrightarrow S$

$$S^n = \underbrace{S \times \dots \times S}_{n \text{ times}}$$

0-ary

$\{\emptyset\} \longrightarrow S$

Monoid

$id_M : M \longrightarrow M$

$a \longmapsto a$

Set M,

0-ary operation : $1 : \{\emptyset\} \longrightarrow M$

2-ary op.. ; $m : M \times M \longrightarrow M$

s.t. $M \times M \times M \longrightarrow M$

$m(id_M \times m) = m(m \times id_M)$

assoc → $m(x, m(y,z)) = m(m(x,y), z) \quad \forall x,y,z \in M$

$x(yz) = (xy)z \qquad xy \equiv m(x,y)$

unit → $m(id_M \times 1) = m(1 \times id_M) = id_M$

$\forall x \in M \quad m(x, 1(\emptyset)) = m(1(\emptyset), x) = x$

$x1 = 1x = x$

$1 \equiv 1(\emptyset)$

$xy \equiv m(x,y)$

Notational Aside

given a product $A \times B$ to define a map

$C \xrightarrow{f} A \times B$

$f(c) = (a,b) \qquad a = "f_1(c)"$

$b = "f_2(c)"$

we write $f = f_1 \times f_2$

Similarly define groups

0-ary op $\quad e: \{\emptyset\} \longrightarrow G \qquad e \equiv e(\emptyset)$

1-ary op $\quad \iota: G \longrightarrow G \qquad g^{-1} \equiv \iota(g)$

2-ary op $\quad m: G \times G \longrightarrow G \qquad gh \equiv m(g,h)$

& same axioms.

Rings $\quad (R, \overbrace{1, 0}^{0\text{ ary}}, \overbrace{\cdot, +}^{2\text{ ary}}, \overbrace{(-)}^{1\text{ ary}})$

<u>Ω-algebras:</u>

Ω a set of symbols w/ "arities"

$$\Omega: \{m, \iota, 1\} \longrightarrow \mathbb{Z}_{\geq 0}$$

$$m \longmapsto 2$$

$$\iota \longmapsto 1$$

$$1 \longmapsto 0$$

<u>Def</u> an Ω-algebra is a set S w/ maps $\lambda: S^n \longrightarrow S$

for each $\lambda \in \Omega$ w/ arity n.

<u>Def</u> homomorphisms of Ω alg's.

are fcns $S \longrightarrow T$ s.t. $\forall \lambda \in \Omega$

$$f(\lambda(s_1, \ldots, s_n)) = \lambda(f(s_1), \ldots, f(s_n))$$

<u>ex:</u>

$$\mathbb{R} \xrightarrow{f} \mathbb{R} \times \mathbb{R}$$

$\searrow 0 \times \mathbb{R}$ or

$$x \longmapsto (0, x)$$

$$x+y \longmapsto (0, x+y) = (0,x)+(0,y)$$

$$xy \longmapsto (0, xy) = (0,x)(0,y)$$

$$1 \longmapsto (0,1) \neq 1$$

$$\lambda = 1 \qquad f(\lambda(\emptyset)) = \lambda(\emptyset)$$

$$f(1) = 1$$

<u>Def</u> (Imprecise) A Variety = the collection of Ω algebras for a given Ω, satisfy a set of identities.

<u>Ex</u>: Ω as above $m, \iota, 1$

w/ identities
$$\left.\begin{aligned} (xy)z &= x(yz) \\ xx^{-1} = x^{-1}x &= 1 \\ 1x = x1 &= x \end{aligned}\right\}$$

Variety defined by these is called "groups"

<u>Fun</u>: <u>Def</u> A congruence on an Ω-algebra A is an Ω-subalgebra of $A \times A$

- Consider G a group $H < G$

$$C = \{(g_1, g_2) \in G \times G \mid g_1 g_2^{-1} \in H\}$$

Show C is a congruence $\iff H \trianglelefteq G$

- Consider a ring R, $I < (R, +)$

$$C = \{(r_1, r_2) \in R \times R \mid r_1 - r_2 \in I\}$$

congruence $\iff I \trianglelefteq R$.

In general, if $A \xrightarrow{f} B$ a hom. of Ω-algs

can define $\ker f = \{(a_1, a_2) \mid f(a_1) = f(a_2)\}$

congruences are kernels.

Doesn't capture all types of structures we care about

<u>Groupoid</u>:

$$G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0$$

$$s(g) \xrightarrow{g} t(g)$$

<u>ex</u>: Pick a collection of sets S

G_1 = bijective maps between these sets.