Nilpotent groups

$\subset = \subseteq$

$\subsetneq$

Recall: G is a group, a sequence of subgroups

$(e) = N_0 \subseteq N_1 \subseteq \dots \subseteq N_d = G$ is an ascending central sequence if

- $N_i \trianglelefteq G$
- $N_{i+1}/N_i \subseteq Z(G/N_i)$

Remark: $xN_i \in Z(G/N_i)$ means $xyx^{-1}y^{-1} \in N_i$ if $y \in G$.

Def $Z_0(G) = (e)$, $Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G))$

i.e. $Z_{i+1}(G)$ is the full preimage of the center of $G/Z_i(G)$

$Z_0(G) \subseteq Z_1(G) \subseteq \dots$ is called the ascending central series of G.

Def G is nilpotent if G has an ascending central sequence.

Lem G is nilpotent iff $Z_d(G) = G$ some d.

Pf: $\Leftarrow$ clear.

$\Rightarrow$ suppose $(e) = N_0 \subseteq \dots \subset N_d = G$ is an asc. central seq.

Claim: $N_i \subseteq Z_i(G)$ all i.

Pf of claim: $i = 0$ ✓

if true for i, then want $N_{i+1} \subset Z_{i+1}(G)$

let $x \in N_{i+1}$, then $xyx^{-1}y^{-1} \in N_i$ all $y \in G$.

$$N_i \subset Z_i \Rightarrow xyx^{-1}y^{-1} \in Z_i(G)$$

$$\Rightarrow xZ_i(G) \in Z\left(G/Z_i(G)\right)$$

$$\Rightarrow x \in Z_{i+1}(G).$$

Proposition: If G is a finite group, the following are equivalent (TFAE)

1) G is nilpotent
2) $\forall H \subsetneq G$, $N_G(H) \supsetneq H$.
3) $P \in Syl_p G \Rightarrow P \triangleleft G$
4) $G = P_1 \dot{\times} \dots \dot{\times} P_r$ P_i's are distinct Sylow subgroups.

Pf: $1 \Rightarrow 2$ ✓ (done before)

$2 \Rightarrow 3$ $P \in Syl_p G$, P char $N_G(P) \triangleleft N_G(N_G(P))$

sublemma: K char $N \triangleleft G \Rightarrow K \triangleleft G$

$$P \triangleleft N_G(N_G(P)) \Rightarrow N_G(N_G(P)) \subset N_G(P)$$
$$\Rightarrow N_G(N_G(P)) \overset{\shortparallel}{} N_G(P)$$
$$\Rightarrow G = N_G(P)$$
$$\Rightarrow P \triangleleft G. \ \square$$

$3 \Rightarrow 4$

If $P_i \triangleleft G$ then claim $P_1 \cdots P_\ell = P_1 \dot{\times} \cdots \dot{\times} P_\ell \triangleleft G$

will induct and. Claim $(P_1 \cdots P_{\ell-1}) P_\ell = (P_1 \cdots P_{\ell-1}) \dot{\times} P_\ell$

need $P_1 \cdots P_{\ell-1} \triangleleft P_1, \ldots, P_\ell$

$P_\ell \triangleleft P_1, \ldots, P_\ell$

$(P_1 \cdots P_{\ell-1}) P_\ell = P_1 \cdots P_\ell$

$(P_1 \cdots P_{\ell-1}) \cap P_\ell = (e)$.

since a product by induction, orders are relatively prime. $\square$.

Remark p-groups are Nilpotent!

if $Z_i(G) \neq G$ then $G/Z_i(G)$ is a non-trivial p-group

$\Rightarrow Z(G/Z_i(G)) \neq \{e\}$ so $Z_{i+1}(G) \supsetneq Z_i(G)$

Enough to show: products of Nilpotent is Nilpotent.

Lem $Z_i(G_1 \times G_2) = Z_i(G_1) \times Z_i(G_2)$

Pf: induct on i. $\quad Z(G_1 \times G_2) = Z(G_1) \times Z(G_2)$

$$
\begin{aligned}
Z_i(G_1 \times G_2)/Z_{i-1}(G_1 \times G_2) &= Z\left(G_1 \times G_2 / Z_{i-1}G_1 \times Z_{i-1}G_2\right) \\
&= Z\left(G_1/Z_{i-1}(G_1) \times G_2/Z_{i-1}(G_2)\right) \\
&= Z\left(G_1/Z_{i-1}(G_1)\right) \times Z\left(G_2/Z_{i-1}(G_2)\right) \\
&= Z_i(G_1)/Z_{i-1}(G_1) \times Z_i(G_2)/Z_{i-1}(G_2)
\end{aligned}
$$

$\varnothing$.

Solvable groups

Def A group G is solvable if $\exists$ subgroups

$$\{e\} = H_n \subset H_{n-1} \subset \cdots \subset H_0 = G \text{ such that}$$

1) $H_{i+1} \triangleleft H_i$

2) H_i / H_{i+1} is Abelian.

"a descending central series"

Lem G solvable, $H < G \Rightarrow H$ solvable,
if $N \triangleleft G$, G/N solvable.

Pf: to see H is solvable, consider $H_i = H \cap G_i$
where $\{e\} = G_n \subset \cdots \subset G_0 = G$

$$H_i \to G_i \longrightarrow G_i / G_{i+1}$$

Abelian

kernel is $H_i \cap G_{i+1}$

$$" (H \cap G_i) \cap G_{i+1} \qquad H_{i+1}$$

$$= H \cap G_i \cap G_{i+1} = H \cap G_{i+1}$$

so $H_i/H_{i+1} \subset G_i/G_{i+1}$ abelian

and $H_{i+1} = \text{kernel}$ so $H_{i+1} \triangleleft H_i$

<u>Prop</u> G solvable, finite $\iff \exists\ (e) = H_n \subset \cdots \subset H_0 = G$

w/ $H_{i+1} \triangleleft H_i$ $\quad H_i/H_{i+1}$ cyclic.

<u>Pf:</u> corresp. theorem & same fact for Abelian gps.

i.e. given H's that have only H_i/H_{i+1} Abelian,

and <u>if</u> can find $(e) = \overline{K}_{i,l_i} \subset \overline{K}_{i,l_i-1} \subset \cdots \subset \overline{K}_{i,0} = H_i/H_{i+1}$

w/ $\overline{K}_{i,j}/\overline{K}_{i,j+1}$ cyclic

by corresp thm: get $K_{i,j}$'s w/

$$H_{i+1} = K_{i,l_i} \subset K_{i,l_i-1} \subset \cdots \subset K_{i,0} = H_i$$

$$K_{i,j}/K_{i,j+1} \cong \overline{K}_{ij}/\overline{K}_{i,j+1}$$

then: $(e) = K_{n,l_n} \subset K_{n,l_n-1} \subset \cdots \subset K_{n,0} = K_{n-1,l_{n-1}} \subset K_{n-1,l_{n-1}-1} \subset \cdots G$

Why can we do this?

i.e. if G is an Abelian gp, want to find subgps

$$(e) = H_n \subset H_{n-1} \subset \cdots \subset H_0 = G \text{ s.t. } H_i/H_{i+1} \text{ cyclic.}$$

Induct on $|G|$.

let $g \in G$, consider $G/\langle g \rangle$ true there.

$$\Rightarrow \quad (e) = \overline{H}_{n-1} \subset \cdots \subset \overline{H}_0 = G/\langle g \rangle \text{ s.t.}$$

$$\overline{H}_i/\overline{H}_{i+1} \text{ cyclic.}$$

by corresp thm, since $H_{n-1} = \langle g \rangle$ we have

$$(e) = H_n \subset H_{n-1} \subset \cdots \subset H_0 = G$$

$$H_i/H_{i+1} \text{ cyclic for } i \neq n-1$$

$$H_{n-1}/H_n = \langle g \rangle \text{ is cyclic.}$$

<u>Def</u> $[G,G]$ is the smallest (normal) subgp containing all commutators $[g,h] = g^{-1}h^{-1}gh$

<u>Note:</u> $hg[g,h] = gh$

Note: $[G,G]$ = smallest subgp containing commutators

$\Leftarrow$ $[G,G]$ char G.

$$\varphi[g,h] = [\varphi(g), \varphi(h)]$$

Rem: if $N \triangleleft G$, then G/N Abelian $\Rightarrow$ $[G,G] \subset N$.

Def $G^{(0)} = G$, $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$

ith derived subgroup.

$$G^{(n)} \triangleleft \dots \triangleleft G^{(2)} \triangleleft G^{(1)} \triangleleft G^{(0)} = G$$

"the descending central series"

Prop G is solvable $\Leftrightarrow$ $G^{(n)} = (e)$ some n.

Pf: if $(e) = H_n \subset H_{n-1} \subset \dots \subset H_0 = G$

$H_{i+1} \triangleleft H_i$ H_i/H_{i+1} Abelian

Claim: $G^{(i)} \subset H_i$

Induction $i=0$ assume true up to i

want $G^{(i+1)} \subset H_{i+1}$

$$G^{(i)} \longrightarrow H_i \longrightarrow H_i/H_{i+1} \text{ Abelian}$$

$$[x,y] \longrightarrow 0$$

$x, y \in G^{(i)}$ $\Rightarrow$ $G^{(i+1)} \subset H_{i+1}$ $\triangleright$.