

Def A (n associative, unital) ring is a set  $R$  w/ binary ops  $\cdot, +$ , distinguished elements  $0, 1$  such that

•  $(R, +, 0)$  is an Ab-grp

•  $(R, \cdot, 1)$  is a monoid

•  $a(x+y) = ax+ay$  ← left distributive law  
all  $x, y, a, b \in R$

$(x+y)b = xb+yb$  ← right dist. law

Ring homomorphisms — corresp.  $\mathcal{L} = (\cdot, +, -, 0, 1)$  algebra  
nom s.

Def A nonassociative, nonunital ring  $R$  is a set  $R$  w/ binary ops  $\cdot, +$ , dist. elmt  $0$  such that

•  $(R, +, 0)$  an Ab-grp

•  $(R, \cdot)$  is a magma

• left & right distributivity

Def A division ring is an (assoc) ring s.t.  $(R \setminus \{0\}, \cdot, 1)$   
is a group  
a commutative division ring is called a field.



$$ab=ba \text{ all } a, b \in \mathbb{R}$$

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## Examples

$k = \text{field}, = \mathbb{Q}, \mathbb{R}, \mathbb{C},$

Hilbert-Division ring

$$\{a+bi+cj+dk \mid a, b, c, d \in \mathbb{R}\}$$

$$i^2 = -1 = j^2 = k^2 \quad ij = k = -ji$$

$\mathbb{Z}$  - important ring.

$$\mathbb{Z}/n\mathbb{Z}$$

$\mathbb{R}$  ring  $X$  set

$$\mathbb{R}^X = \{f: X \rightarrow \mathbb{R}\}$$

ring structure

$$(f+g)(x) = f(x) + g(x)$$

$$(f \cdot g)(x) = f(x)g(x)$$

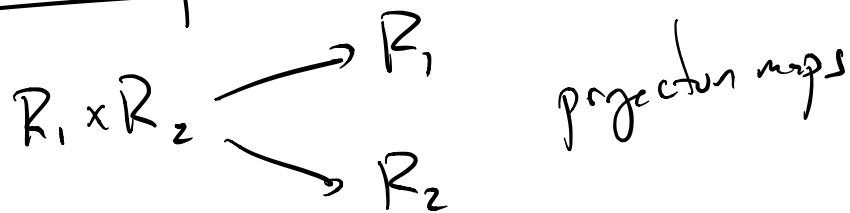
$$\cong \prod_{x \in X} \mathbb{R}$$

$$R_1 \times R_2 = \{(r_1, r_2) \mid r_i \in R_i\}$$

$$(r_1, r_2) + (r_1', r_2') = (r_1 + r_1', r_2 + r_2')$$

$$(r_1, r_2) \cdot (r_1', r_2') = (r_1 \cdot r_1', r_2 \cdot r_2')$$

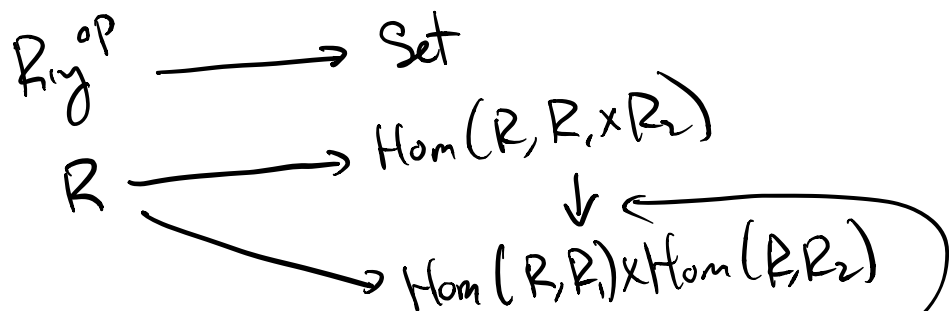
Characterization of product



$$\text{Hom}_R(R, R_1 \times R_2) \longrightarrow \text{Hom}_R(R, R_1) \times \text{Hom}_R(R, R_2)$$

fix  $R_1, R_2$

functors



$R_1 \times R_2$  characterized by being a natural isom. of

functrs.

Def  $R$  is a ring, we say  $r \in R \setminus \{0\}$  is a zero-divisor if  
 $rx=0$  or  $xr=0$  some  $x \neq 0$

we say  $r \in R$  is a unit if  $\exists u \in R$  s.t.  
 $ur = ru = 1$

my example  $R[x] = \left\{ \sum_{i=0}^n a_i x^i \mid a_i \in R \right\}$

$$(ax^i)(bx^j) = abx^{i+j}$$

$$R[[x]] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in R \right\}$$

$$\left( \sum a_i x^i \right) \left( \sum b_j x^j \right) = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} a_i b_j \right) x^k$$

~~$$R[[x^{-1}, x]] = \sum_{i \in \mathbb{Z}} a_i x^i$$~~

$$R((x)) = \left\{ \sum_{i \geq 0} a_i x^i \right\} \text{ same mult.}$$

$d \in \mathbb{Z}$

exercise: if  $R$  is a field, so is  $R[x]$

in  $\mathbb{Z}$ , only units are  $\pm 1$

no zero-divisors

Def A commutative ring is called a domain if it has no zero divisors and  $1 \neq 0$

Def The zero ring is the ring with one element  $1=0$

If  $R$  is any ring,  $\exists!$  ring homomorphism  $\mathbb{Z} \rightarrow R$

and  $\exists!$  ring hom  $R \rightarrow 0$

Def If  $\mathcal{C}$  is a category, an object  $x$  is called initial if  $\forall$  objects  $z$ ,  $\exists!$  element in  $\text{Hom}(x, z)$   
it is terminal if  $\exists!$  element in  $\text{Hom}(z, x)$

If  $\mathcal{C}$  is a category,  $x, y$  objects,  $f \in \text{Hom}_{\mathcal{C}}(x, y)$  is an iso  
if  $\exists g \in \text{Hom}_{\mathcal{C}}(y, x)$  st.  $fg = 1_y$ ,  $gf = 1_x$

Exercise If  $R$  is a domain, so is  $R[x]$

If  $M$  is any monoid,  $R$  any, can form

$$\text{monoid-rng } R[M] = \left\{ \sum_{i=1}^n r_i m_i \mid m_i \in M, r_i \in R \right\}$$

$$(rm)(sn) = (rs)(mn)$$

typically (potentially if  $M$  is commutative)  
convenient to write  $x^m$  for  $m$

$$\sum r_i x^{m_i}$$

$$\begin{aligned} (rx^m)(sx^n) \\ = rsx^{m+n} \end{aligned}$$

example:  $M = (\mathbb{Z}_{\geq 0}, +)$   $R[M] = R[x]$

if  $M$  is a group, this is usually called the  
group algebra.

example  $R[\mathbb{Z}] = R[x, x^{-1}]$

$$\text{Hom}_{\text{rng}}(\mathbb{Z}[x], R) = R \text{ as a set.}$$

$$\begin{array}{ccc} \mathbb{Z}[x] & \xrightarrow{\varphi} & R \\ x & \longmapsto & r \end{array}$$

$$\begin{array}{ccc} f(x) & \longmapsto & f(r) \\ \text{"} & & \text{"} \\ \sum a_i x^i & \longmapsto & \sum a_i r^i \end{array}$$

$$\varphi(f+g) = \varphi(f) + \varphi(g)$$

$$\varphi(fg) = \varphi(f)\varphi(g)$$

$$\text{Hom}_{\text{rng}}(\mathbb{Z}[x, y], R) = \left\{ (a, b) \in R \times R \mid a, b \text{ commute} \right\}$$

$$\text{Hom}_{\text{rng}}(\underbrace{\mathbb{Z}[\mathbb{Z}]}_{\mathbb{Z}[x, x^{-1}]}, R) = R^* = \{ \text{units in } R \}$$

ex  
 $M$  a monoid  $\text{Hom}_{\text{rng}}(\mathbb{Z}[M], R) = \text{Hom}_{\text{monoid}}(M, R)$

These have the shape

Rings  $\longrightarrow$  Sets

$\mathbb{R} \longrightarrow \mathbb{R}$

$\mathbb{R} \longrightarrow \{\text{pairs of commuting divisors in } \mathbb{R}^*\}$

$\mathbb{R} \longrightarrow \mathbb{R}^*$

$\mathbb{R} \longrightarrow \{\text{monoid homs } M \rightarrow \mathbb{R}^*\}$   
(fixed  $M$ )

We have shown that these functors "are" all of the form  $\text{Hom}_{\text{ring}}(T, -)$  for various  $T$  ( $T = \mathbb{Z}\langle x \rangle$ ,  $\mathbb{Z}\langle x, y \rangle$ , ...)

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Given cats  $\mathcal{C}, \mathcal{D}$ , consider  $\text{Fun}(\mathcal{C}, \mathcal{D})$

the category of functors from  $\mathcal{C}$  to  $\mathcal{D}$

$\text{ob}(\text{Fun}(\mathcal{C}, \mathcal{D})) = \text{functors from } \mathcal{C} \text{ to } \mathcal{D}$

$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathcal{D})}(F, G) = \left\{ \begin{array}{l} \text{natural transformations} \\ \eta: F \Rightarrow G \end{array} \right\}$



All the above functors are in  $\text{Fun}(\mathcal{C}, \text{Set})$

$\mathcal{C} = \text{Rys}$

given any object  $T$  in  $\mathcal{C}$ , get such a functor

$$\begin{array}{ccc} & \text{Hom}_{\mathcal{C}}(T, -) & \\ \text{gives} & \rho^{\circ\rho} \longrightarrow & \text{Fun}(\mathcal{C}, \text{Set}) \\ & T \longmapsto & \text{Hom}_{\mathcal{C}}(T, -) \end{array}$$

Theorem (Yoneda Lemma) The above functor  
is fully faithful

( if  $F: \mathcal{C} \rightarrow \mathcal{D}$ , given  $x, y$   $\text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{F} \text{Hom}_{\mathcal{D}}(F_x, F_y)$   
if these injective  
"faithful"  
if these surjective  
"full" )