<u>Def</u> A(n associative, unital) ring is a set R w/ binary ops $\cdot, +$, distinguished elements $0, 1$ such that

- $(R, +, 0)$ is an Ab. gp
- $(R, \cdot, 1)$ is a monoid
- $a(x+y) = ax + ay$ left distributive law

  all $x, y, a, b \in R$

  $(x+y)b = xb + yb$ right dist. law

Ring homomorphisms – corresp. $\Omega = (\cdot, +, -, 0, 1)$ algebra homs.

<u>Def</u> A nonassociative, nonunital ring R is a set R w/ binary ops $\cdot, +$, dist. elmt 0 such that

- $(R, +, 0)$ an Ab. gp
- $(R, \cdot)$ is a magma
- left & right distributivity

<u>Def</u> A division ring is an (assoc. unital) ring s.t. $(R \setminus \{0\}, \cdot, 1)$ is a group

a commutative division ring is called a field.

$ab = ba$ all $a, b \in R$

__Examples__

k = field, $= \mathbb{Q}, \mathbb{R}, \mathbb{C}$,

$\mathbb{H}$ - division ring

$= \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$

$i^2 = -1 = j^2 = k^2 \quad ij = k = -ji$

$\mathbb{Z}$ - important ring.

$\mathbb{Z}/n\mathbb{Z}$

R ring $\quad X$ set $\qquad R^X = \{f : X \to R\}$

ring via

$(f+g)(x) = f(x) + g(x)$

$(f \cdot g)(x) = f(x)g(x)$

$$\cong \prod_{x \in X} R$$

$$R_1 \times R_2 = \{(r_1, r_2) \mid r_i \in R_i\}$$

$$(r_1, r_2) + (r_1', r_2') = (r_1 + r_1', r_2 + r_2')$$

$$(r_1, r_2) \cdot (r_1', r_2') = (r_1 \cdot r_1', r_2 \cdot r_2')$$

Characterization of product:

$$\begin{tikzcd} & R_1 \\ R_1 \times R_2 \arrow[ur] \arrow[dr] & \\ & R_2 \end{tikzcd}$$

projection maps

$$\mathrm{Hom}_{\mathrm{Ring}}(R, R_1 \times R_2) \longrightarrow \mathrm{Hom}_{\mathrm{Ring}}(R, R_1) \times \mathrm{Hom}_{\mathrm{Ring}}(R, R_2)$$

fix R_1, R_2

functors

$$\mathrm{Ring}^{op} \longrightarrow \mathrm{Set}$$

$$R \longrightarrow \mathrm{Hom}(R, R_1 \times R_2)$$

$$\downarrow$$

$$R \longrightarrow \mathrm{Hom}(R, R_1) \times \mathrm{Hom}(R, R_2)$$

$R_1 \times R_2$ characterized by

by a natural isom. of

functors.

Def R is a ring, we say $r \in R \setminus \{0\}$ is a zero-divisor if

$rx = 0$ or $xr = 0$ some $x \neq 0$

we say $r \in R$ is a unit if $\exists\, u \in R$ s.t.

$ur = ru = 1$

ring example

$$R[x] = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in R \right\}$$

$$(ax^i)(bx^j) = ab\, x^{i+j}$$

$$R[[x]] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in R \right\}$$

$$\left(\sum a_i x^i\right)\left(\sum b_j x^j\right) = \sum_{k=0}^{\infty} \left(\sum_{i+j=k} a_i b_j\right) x^k$$

$$\cancel{R[[x^{-1}, x]] = \sum_{i \in \mathbb{Z}} a_i x^i}$$

$$R((x)) = \left\{ \sum_{i \geq d}^{\infty} a_i x^i \right\} \quad \text{same mult.}$$

$d \in \mathbb{Z}$

exercise: if R is a field, so is $R((x))$

in $\mathbb{Z}$, only units are ± 1

no zero-divisors

Def A commutative ring is called a domain if it has no zero divisors and $1 \neq 0$

Def The zero ring is the ring w/ one element $1=0$

If R is any ring, $\exists!$ ring homomorphism $\mathbb{Z} \to R$

and $\exists!$ ring hom $R \longrightarrow 0$

Def If $\mathcal{C}$ is a category, an object x is called initial if $\forall$ objects z, $\exists!$ element in $\mathrm{Hom}(x,z)$

it is terminal if $\exists!$ element in $\mathrm{Hom}(z,x)$

If $\mathcal{C}$ is a category, x,y objects, $f \in \mathrm{Hom}_{\mathcal{C}}(x,y)$ is an iso if $\exists$ $g \in \mathrm{Hom}_{\mathcal{C}}(y,x)$ s.t. $fg = 1_y$, $gf = 1_x$

__Exercise__ If R is a domain, so is $R[x]$

If M is any monoid, R a ring, can form monoid-ring $R[M] = \left\{ \sum_{i=1}^{n} r_i m_i \;\middle|\; m_i \in M, r_i \in R \right\}$

$$(rm)(sn) = (rs)(mn)$$

typically (particularly if M is commutative) conventional to write x^m for m

$$\sum r_i x^{m_i} \qquad (rx^m)(sx^n) = rsx^{m+n}$$

__example__: $M = (\mathbb{Z}_{\geq 0}, +)$ $\quad R[M] = R[x]$

if M is a group, this is usually called the __group algebra.__

__example__ $R[\mathbb{Z}] = R[x, x^{-1}]$

$$\mathrm{Hom}_{rng}(\mathbb{Z}[x], R) = R \text{ as a set.}$$

$$\mathbb{Z}[x] \xrightarrow{\varphi} R$$

$$x \longmapsto r$$

$$f(x) \longmapsto f(r)$$

$$\begin{array}{ccc} f(x) & \longmapsto & f(r) \\ \| & & \| \\ \sum a_i x^i & \longmapsto & \sum a_i r^i \end{array}$$

$$\varphi(f+g) = \varphi(f) + \varphi(g)$$

$$\varphi(fg) = \varphi(f)\varphi(g)$$

$$\mathrm{Hom}_{rng}(\mathbb{Z}[x,y], R) = \{(a,b) \in R \times R \mid a, b \text{ commute}\}$$

$$\begin{array}{l} \mathrm{Hom}_{rng}(\mathbb{Z}[\mathbb{Z}], R) = R^* \equiv \{\text{units in } R\} \\ \quad\quad\quad\quad\; \| \\ \quad\quad\quad \mathbb{Z}[x, x^{-1}] \end{array}$$

<u>ex</u>

M a monoid $\quad \mathrm{Hom}_{rng}(\mathbb{Z}[M], R) = \mathrm{Hom}_{monoid}(M, R)$

These have the shape

$$\text{Rings} \longrightarrow \text{Sets}$$
$$R \longmapsto R$$
$$R \longmapsto \{\text{pairs of commuting elements in } R\}$$
$$R \longmapsto R^{\times}$$
$$R \longmapsto \{\text{monoid homs } M \to R\}$$
$$(\text{fixed } M)$$

We have shown that these functors "are" all of the form $\mathrm{Hom}_{\text{ring}}(T, -)$ for various T ($T = \mathbb{Z}[x]$, $\mathbb{Z}\langle x, y\rangle$, ...)

Given cats $\mathcal{C}, \mathcal{D}$, consider $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$

$\mathrm{ob}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$ = functors from $\mathcal{C}$ to $\mathcal{D}$

$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G) = \{\text{natural transformations } \eta : F \Rightarrow G\}$

All the above functors are in $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$

$\mathcal{C} = \mathrm{Rngs}$

given any object T in $\mathcal{C}$, get such a functor

$\mathrm{Hom}_{\mathcal{C}}(T, -)$

gives
$$\begin{aligned} \mathcal{C}^{op} &\longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Set}) \\ T &\longmapsto \mathrm{Hom}_{\mathcal{C}}(T, -) \end{aligned}$$

<u>Theorem</u> (Yoneda Lemma) The above functor is fully faithful

(if $F : \mathcal{C} \to \mathcal{D}$, given x, y $\quad \mathrm{Hom}_{\mathcal{C}}(x,y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(Fx, Fy)$

if these injective "faithful"

if these surjective "full")