R integral domain $\operatorname{frac}(R)$ $\operatorname{quo}(R)$ denote frac. field

$$\operatorname{frac}(R) = \left\{ \frac{a}{b} \;\middle|\; a,b \in R,\ b \neq 0 \right\}$$

$$\frac{a}{b} = \frac{c}{d} \text{ if } ad = bc$$

Ex: $\mathbb{Z}[i] = R = \{a+bi \mid a,b \in \mathbb{Z}\} \subset \mathbb{C}$

$$\operatorname{frac} \mathbb{Z}[i] = \left\{ \frac{a+bi}{c+di} \;\middle|\; a,b \ldots \right\} \text{ w/ complicated addition \& mult.}$$

$$\mathbb{Q}[i] = \mathbb{Q} \oplus \mathbb{Q}i \subset \mathbb{C}$$

check: this a field

$$(a+bi)\left(\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i \right) = 1$$

$\mathbb{Q}[i]$ field, contains $\mathbb{Z}[i]$

$$\begin{tikzcd}
\mathbb{Z}[i] \arrow[rr] \arrow[dr] & & \mathbb{Q}[i] \\
& \operatorname{frac} \mathbb{Z}[i] \arrow[ur, "\exists!"'] &
\end{tikzcd}$$

$$\frac{z}{w} \longmapsto z w^{-1}$$

so $\operatorname{frac} \mathbb{Z}[i] \to \mathbb{Q}[i]$

ker $= 0$ since no nontrivial ideals, not 0 map since $1 \mapsto 1$.

$\Rightarrow \cong$!

<u>Moral</u>: Abstract localization useful because it exists. In practice, one often can do better.

<u>Ex</u> $R = \mathbb{C}[\![x]\!]$ $\quad \operatorname{frac} R = \left\{ \frac{f}{g} \right\}$

$\mathbb{C}((x)) = \left\{ \sum_{i \geq d} a_i x^i \right\}$ is a field containing R.

why field?

1. if $f(x) \in x\,\mathbb{C}[\![x]\!]$

$$\left(\sum_{i=0}^{\infty} f(x)^i \right)(1 - f(x)) = 1$$

<u>note</u>: $\mathbb{C}^* \oplus x\,\mathbb{C}[\![x]\!] = \mathbb{C}[\![x]\!]^*$

2. if $f(x) = \sum_{i \geq d} a_i x^i$ then

$$f(x) = x^{-d} a_d^{-1} (1 - g(x))$$

$$\begin{tikzcd} \mathbb{C}[[x]] \arrow[rr] \arrow[dr] & & \mathbb{C}((x)) \\ & \operatorname{frac} \mathbb{C}[[x]] \arrow[ur, "\sim"] & \end{tikzcd}$$

$x^{-d} f(x) = \dfrac{f(x)}{x^d}$ $f(x) \in \mathbb{C}[[x]]$

"Partial denominators"

$$\mathbb{Z}[1/5] = \left\{ \frac{a}{5^n} \;\middle|\; a \in \mathbb{Z},\ n \geq 0 \right\} \subset \mathbb{Q}$$

$$\mathbb{Z}[1/5, 1/3] = \mathbb{Z}[1/15]$$

$$\mathbb{C}[x,y][x^{-1}] = \left\{ \frac{f(x,y)}{x^n} \right\}$$

$$\mathbb{C}[x,y][x^{-1}, (y-1)^{-1}]$$

$R[S^{-1}]$ $(0 \notin S)$ $\mathbb{Z}[1/3] = \mathbb{Z}[\{1, 3, 3^2, \dots\}^{-1}]$

if R is a domain, this means the subring of $\operatorname{frac}(R)$ consisting of $\left\{ \frac{a}{b} \in \operatorname{frac} R \;\middle|\; b \in S \right\}$

S = submonoid of $(R, \cdot)$

If R not a domain, but S consists of regular elements

$R[S^{-1}] = \left\{ \frac{a}{b}, b \in S \right\}$ same rules as before.

= nonzero, non-zerodivisor

<u>Def</u> R commutative ring, $r \in R$ is regular if $r \neq 0$ and $rs = 0 \Rightarrow s = 0$.

What if $s \in S$ is a zero divisor?

want $R \longrightarrow R[S^{-1}]$

elements of S map to units in $R[S^{-1}]$

if $sx = 0$ then in $R[S^{-1}]$ need to have $\bar{x} = 0$

$\bar{s}\bar{x} = 0$

$\bar{s}^{-1}\bar{s}\bar{x} = 0$

$\bar{x} = 0$

$I = \{ r \in R \mid rs = 0 \text{ some } s \in S \}$

$\bar{S} = \{ s + I \mid s \in S \} \in R/I$

$R[S^{-1}] \cong R/I\,[\bar{S}^{-1}]$

$R[S^{-1}] = \left\{ \frac{a}{b} \;\middle|\; a \in R, b \in S \right\}$ same op.

$\frac{a}{b}$ eq. class of $(a,b) \in R \times S$ w/r/to e. relation

$(a,b) \sim (a',b')$

$t(ab' - a'b) = 0 \quad t \in S$

Chinese Remainder Theorem

R comm. ring $I, J \triangleleft R$, we say I & J are comaximal if

$I + J = R$.

<u>Note</u>: If $I, J \triangleleft R$ comaximal then

$I \cap J = IJ$.

<u>Pf</u>: $IJ \subset I \cap J$ ✓

$I \cap J = (I \cap J)R = (I \cap J)(I + J)$

$= (I \cap J)I + (I \cap J)J$

$\subset JI + IJ = IJ$ ✓

Theorem (CRT) if $I, J \trianglelefteq R$ comaximal

$$R/IJ \simeq R/I \times R/J$$

Pf since $R = I + J$, can write $1 = e + f$
$e \in I$, $f \in J$.

consider $\bar{f}$ in R/I and $\bar{e} \in R/J$

$$\begin{aligned} R &\longrightarrow R/I,\ R/J \\ 1 &\longmapsto \overline{e+f},\ \overline{e+f} \\ &\quad \bar{f} = \bar{e} + \bar{f} \qquad \bar{e} + \bar{f} = \bar{e} \end{aligned}$$

so $f \longmapsto 1$ in R/I & 0 in R/J

$e \longmapsto 1$ in R/J & 0 in R/I

$$\begin{aligned} R &\longrightarrow R/I \times R/J \\ xf + ye &\longmapsto (\bar{x}, \bar{y}) \end{aligned}$$

so surjective.

kernel?

$r \in R$ in kernel iff $r \in I$ & $r \in J$

$\ker = I \cap J = IJ$. ∎

$\mathbb{Z}/n\mathbb{Z}$ $\quad n = ab \quad (a,b) = 1 \quad a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$

$\cong \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$

similarly $F[x] = R \quad (f,g) = 1$

$fR + gR = R$

$$F[x]/fg \cong F[x]/f \times F[x]/g.$$

<u>Principal Ideal Domains</u>

<u>Def</u> A ideal $I \triangleleft R$ is <u>principal</u> if $I = (a)$, some $a \in R$.

<u>Def</u> A commutative domain R is a PID iff all ideals $I \triangleleft R$ are principal.

Ex $\mathbb{Z}$ (and other Euclidean domains)

(if $I \lhd \mathbb{Z}$ can choose $d \in I$ w/ $|d|$ minimal.
if $n \in I$, $(d,n) = xd + yn$, we have
$(d,n) \in I$, but $|(d,n)| \leq |d|$ so $=$.
$\Rightarrow (d,n) = \pm d \Rightarrow n \in d\mathbb{Z}. \Rightarrow I = d\mathbb{Z}$)

Ex: $F[x]$ F a field.

Prop If R is a PID, all prime ideals of R are maximal.

Pf: suppose $\overset{\neq 0}{\mathfrak{P}} \lhd R$ prime, $\mathfrak{P} = pR$
suppose $\mathfrak{P} \lneq I \lhd R$. write $I = mR$

so $p = mr$ some r $mr = p \in \mathfrak{P} \Rightarrow m \in \mathfrak{P}$ or $r \in \mathfrak{P}$.

if $m \in \mathfrak{P} \Rightarrow \underset{I}{\underset{\shortparallel}{mR}} \subset \mathfrak{P} \Rightarrow \mathfrak{P} = I$.

$p(1-ms) = 0$

if $r \in \mathfrak{P} \Rightarrow r = ps$ $p = mr = msp$

R domain $\Rightarrow ms = 1$
$\Rightarrow m \in R^*$

$I = mR = R$. $\square$.

<u>Cor</u> $R[x]$ is a PID $\Longleftrightarrow$ R a field.

<u>Pf</u> $\Leftarrow$ ✓

$\Rightarrow$ let $I = xR[x]$ $\quad R[x]/I \cong R$

$R[x]$ PID $\Rightarrow$ a domain

$R \subset R[x] \Rightarrow R$ a domain.

$R[x]/I$ domain $\Rightarrow I$ prime.

$\Rightarrow I$ max'l $\quad \underset{\substack{\cong \\ R}}{R[x]/I}$ field