Correspondence theorem for localization

R commutative ring, $S \subset R$ multiplicative set then $\exists$ bijective correspondence

$$\{\text{ideals of } R[S^{-1}]\} \longleftrightarrow \{\text{ideals of } R \text{ disjoint from } S\}$$

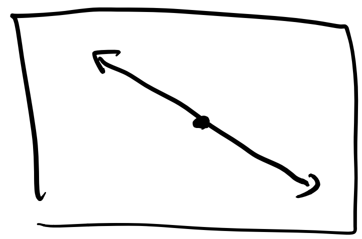
if $\varphi : R \to R[S^{-1}]$

$$I \longmapsto \varphi^{-1}(I)$$

$$JR[S^{-1}] \longleftarrow\!\shortmid J$$

Some rings

$\mathbb{C}[x] \longleftrightarrow$

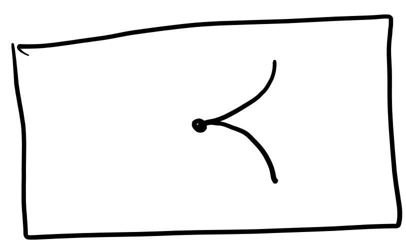


$\mathbb{C}[x,y]/(x+y) \simeq \mathbb{C}[x]$

$x \mapsto x$

$y \mapsto -x$

$R = \mathbb{C}[x,y]/(y^2 - x^3)$



$\bar{m} = (\bar{x}, \bar{y})$ (i.e. $(x,y)/(y^2-x^3)$)

Claim: m is not principal.

Note: $\mathbb{C}[t^2, t^3] \subset \mathbb{C}[t]$

$t^2 \leftrightarrow x$, $t^3 \leftrightarrow y$ — $\mathbb{C}[x,y]/y^2-x^3$

If it was, $\exists \bar{f} \in \bar{m}$ s.t. $\bar{m} = (\bar{f})$

this would mean: $m = (x,y) = (f, y^2 - x^3)$

look in $R/\bar{m}^2 = R/(\bar{x}^2, \bar{x}\bar{y}, \bar{y}^2)$

$= \mathbb{C}[x,y]/m^2 = \mathbb{C}[x,y]/(x^2, xy, y^2)$

$\bar{m}, (\bar{f}) \subseteq R \rightsquigarrow$ images in $R/\bar{m}^2$

$(\tilde{f}), \tilde{m}$ in $R/\bar{m}^2 = \mathbb{C}[x,y]/m^2$

$\tilde{m} = \mathbb{C}\tilde{x} + \mathbb{C}\tilde{y}$ $\qquad f = ax + by + \text{HOT's}$

2 dim'l/$\mathbb{C}$ $\qquad (\tilde{f}) = \mathbb{C}(ax+by) \leftarrow$ 1-dim'l/$\mathbb{C}$

Def R commutative domain. Let $r \in R \setminus \{0\}$, $r \notin R^{*}$.

- We say that r is irreducible if $r = ab \Rightarrow a$ or b is in R^{*}.
- We say that r is prime if (r) is prime. i.e. $r \mid ab \Leftrightarrow r \mid a$ or $r \mid b$
- We say that $a, b \in R \setminus \{0\}$, nonunits are associate if $a = bu$ some $u \in R^{*}$.

Def R is a UFD if $\forall r \in R \setminus \{0\} \cup R^{*}$, we can write $r = p_1 \cdots p_n$ p_i irreducible & p_i unique up to permutation & associates.

Rem prime $\Rightarrow$ irred

p prime, $p = ab \Rightarrow ab \in (p)$ say $a \in (p)$

$\Rightarrow p = ab = pcb \Rightarrow bc = 1$ $b \in R^{*}$ $\square$.

but irred $\not\Rightarrow$ prime in general

Rem if R is a UFD, irred $\Rightarrow$ prime.

Pf: if $p \in R$ irred, suppose $p \mid ab \Rightarrow pc = ab$

$p \cdot c_1 \cdots c_i = a_1 \cdots a_j b_1 \cdots b_k$

$\Rightarrow p \mid$ some a_i or b_i

$\Rightarrow\Leftarrow$.

if $r = ab = cd$ different factorizations into irreds

then $(a) \ni cd$ if $c \in (a)$ then

$c = ar \Rightarrow$ irred a appr in c is dumb

<u>ex</u>: $\mathbb{Z}[\sqrt{-5}]$ $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

<u>Thm</u>☆ R domain. R UFD $\Longleftrightarrow R[x]$ UFD

<u>most important ys</u>: $\mathbb{Z}[x_1, \ldots, x_n]/I$

$\mathbb{Q}\{\quad\}/I$

$\mathbb{C}[\quad]/I$

<u>Lemma</u> (Gauss' Lemma) Let R be a UFD.

$f(x) \in R[x]$. Let $F = \mathrm{frac}(R)$.

If $f(x)$ factors in $F[x]$ as $f(x) = p(x) q(x)$

then can factor $f(x)$ in $R[x]$ as $\tilde{p}(x)\tilde{q}(x)$

where $\tilde{p}(x) = a\,p(x)$, $\tilde{q}(x) = b\,q(x)$, $a, b \in F$.

<u>Lemma</u> R comm. ring, $I \triangleleft R$ prime $\Leftrightarrow$ $I[x] \triangleleft R[x]$ prime
$I[x] = IR[x]$

<u>Pf:</u> R/I domain $\Leftrightarrow$ $(R/I)[x]$ a domain

$$R[x] \longrightarrow (R/I)[x]$$

kernel is $I[x]$.

<u>Pf of Gauss' lemma</u>

Suppose $f \in R[x]$, $f = pq$, $p, q \in F[x]$.

Clear denominators i.e. $p = \sum p_i x^i$ $q = \sum q_i x^i$

$$q_i = \frac{c_i}{d_i} \quad p_i = \frac{a_i}{b_i} \quad d = \left(\prod d_i\right)\left(\prod b_i\right)$$

get $df = p'q'$

So in other words, we know can factor

$$\lambda f = \tilde{p}\tilde{q} \quad \tilde{p}, \tilde{q} \text{ mults of } p, q, \lambda \in R.$$

choose such a presentation with λ having a minimal # of irred factors.

Want: it's a unit.

If not, say π factor of λ.
$\qquad\qquad\quad$ ^irred.

consider $\tilde{p}\tilde{q} = \lambda f$ in $R[x]/\pi R[x] = \left(R/(\pi)\right)[x]$

RHS $\to 0 \Rightarrow \tilde{p}\tilde{q} = 0$ in this ring

π irred R UFD $\Rightarrow \pi$ prime $\Rightarrow$
$\pi R[x]$ prime. $\Rightarrow$ domain

say $\tilde{p} \in \pi R[x]$. $\Rightarrow$ each coeff of $\tilde{p}$ divisible by π

$$\lambda f = \pi \lambda' f = \pi \tilde{\tilde{p}} \tilde{q}$$

$$\lambda' f = \tilde{\tilde{p}} \tilde{q}$$

λ' one less factor ~~λ~~.

$\square$.

Proof of Theorem ☆

If $R[x]$ is a UFD $\Rightarrow$ get factorizations for $r \in R \subset R[x]$ into irreds in $R[x]$
$\Rightarrow$ gives a factorization in R. uniqueness from $R[x]$
(note: $R[x]^* = R^*$)

For the converse, suppose R is a UFD.

Suppose $f(x) \in R[x]$. Let $d = \gcd$ of coeffs of $f(x)$

($\gcd\{a_1, \ldots, a_n\} = d$ means $d | a_i$ all i & $e | a_i$ all $i \Rightarrow e | d$)

there exist via $d = \prod$ common irreds in a_j's.

$f(x) = d\, g(x)$

consider $g(x)$ in $F[x]$ $\overset{\text{PID}}{\Rightarrow}$ UFD, can write

$g(x) = g_1(x) \cdots g_n(x)$ in $F[x]$ irreds g_i

by Gauss, can assume $g_i(x) \in R[x]$.

Claim $\gcd\{$coeffs of $g_i\} = 1$ all i. ✓

Unique?

$d\, g_1 \cdots g_n = d'\, h_1 \cdots h_m$ in $F[x]$ $g_i = h_j$ up to perm & assoc.

$d\, g_1 \cdots g_n = d'\, h_1 \cdots h_n$

$g_i = \lambda_i h_i$ $\lambda_i = \dfrac{a_i}{b_i}$

$b_i g_i = a_i h_i$ $a_i, b_i \in R$

WLOG, a_i, b_i have no irred factors in common
(could cancel in expression for λ_i)

if p factor of a_i then $p \nmid b_i \Rightarrow p \mid g_i$ contradicting
fact that gcd{coeffs of g_i} = 1.

$\Rightarrow$ WLOG $a_i = 1$ $\quad \lambda_i = \frac{1}{b_i}$

$b_i g_i = h_i \Rightarrow h_i$ not irred in $R[x]$.

(unless b_i unit) $\quad \cancel{\#}$. $\quad \square$.