Midterm: $\boxed{\text{November 14}}$

Def A commutative ring R is Noetherian if every ideal $I \triangleleft R$ is finitely generated (i.e. $I = \langle a_1, \ldots, a_n \rangle$)

(similarly, if R not necessarily comm, can define right-Noetherian & left-Noetherian)

From now on, R commutative

Remark If R is Noeth, $I \triangleleft R$ then R/I also Noetherian.

(if $\bar{J} \triangleleft R/I$ $\quad \bar{J} = J/I \quad J = \langle a_1, \ldots, a_n \rangle$ in R

$\bar{J} = \langle \bar{a}_1, \ldots, \bar{a}_n \rangle$ in R/I)

Thm (Hilbert Basis Theorem) If R is Noeth, so is $R[x]$.

Pf: Let $I \triangleleft R[x]$

$f(x) \in R[x]$

$f(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots$

a_d = leading term (if $a_d \neq 0$)

Observation: if $J = \{\text{leading terms of elmts of } I\} \cup \{0\}$

then $J \triangleleft R$.

$f(x) = ax^d + LOT \qquad g(x) = bx^e + LOT$

ra = leading term of $rf(x)$ (unless $=0$)

in any case $ra \in J$

$x^e f + x^d g$ has leading term $a+b$.

$a+b \in J$. (unless 0)

$\Rightarrow$ (R Noeth) can choose finite set $f_1, \dots, f_N$ whose leading terms generate J.

$\in I$

let $D = \max\{\text{degree } f_i\}$.

consider $\tilde{I} = \langle f_1, \dots, f_N \rangle \subset I$

Claim: if $g \in I$, then $\exists f \in \tilde{I}$ s.t.

$g - f$ has degree $\leq D$.

Pf: by induction on degree of g.

if $\deg g \le D$ ✓

assume true if $\deg g \le m$

suppose $\deg g = m+1$

$g = a x^{m+1} + LOT \quad a \in J$

$f_i = a_i x^{d_i} + LOT$ know $d_i \le m+1$

$a \in \langle a_i \rangle \quad a = \sum b_i a_i \quad b_i \in R$

$g - \underbrace{\sum x^{m+1-d_i} b_i f_i}_{\alpha \in \tilde{I}}$ has degree at most m

$g - \alpha \in I$ $\deg \le m$ $\quad g - \alpha - f'$ $\deg \le D$ for some $f' \in \tilde{I}$

set $f = \alpha + f'$ ✓

Observation: for any d, can find a finite collection of poly's of degree d in I

$\mathcal{G}_d = \{g_1, \dots, g_\ell\}$ such that if $f \in I$ of $\deg d$ then can find $r_i \in R$ s.t. $f - \sum r_i g_i$ $\deg d-1$

if true, then $\{f_1, \ldots, f_N\} \cup \bigcup_{d \le D} \mathcal{G}_d$ generates I.

Why can we find $\mathcal{G}_d$'s?

let $J_d = \{$leading terms of elements of I which have deg $d\} \cup \{0\}$

$J_d \trianglelefteq R$

L.term$(f+g)$ = l.term f + l.term g unless 0.

l.term(rf) = $r \cdot$ leading term(f) unless 0.

J_d f. generated, choose $\{g_1, \ldots, g_\ell\}$ in I of deg d whose l.terms gen J_d.

$\square$

Modules!

Def R ring, a (unital left) R-module is a Abelian group M w/ operation $R \times M \to M$ s.t.
$(r,m) \mapsto rm$

- $r(sm) = (rs)m$ all $r, s \in R$, $m \in M$
- $r(m+n) = rm + rn$ all $r \in R$, $m, n \in M$
- $1 \cdot m = m$ all $m \in M$

• $(r+s)m = rm + sm$ all $r,s \in R$, $m \in M$.

Similarly, right modules are . . .

<u>Ex</u>: $R = C^\infty_{\mathbb{R}}(X)$ $X = \mathbb{R}^n$

M = vector fields on X

<u>Ex</u>: V a vector space / F - dimension n.

$\mathrm{End}_F(V)$ linear transformations

V an F-module, also $\mathrm{End}_F(V)$-module.

<u>Def</u> If V an R-module, $W \subset V$ is a submodule if it is a subgroup s.t. $RW \subset W$ (i.e. closed under ops.)

<u>Def</u> If V, W R-modules, an (R-module) (homo)morphism is an ab. gp hom. $V \xrightarrow{\varphi} W$ st. $\varphi(rv) = r\varphi(v)$ all $r \in R$, $v \in V$.

<u>Remark</u>: if M is an R-module,

$\mathrm{End}_R M = \{f : M \to M \mid f \text{ an } R\text{-mod hom}\}$

is a ring. w/ mult. given by composition
&, pointwise addition

$$(f+g)(m) = f(m)+g(m)$$

$$(fg)(m) = f(g(m))$$

and M is a <u>left</u> $End_R(M)$-module
(even if M is a right R-mod)

<u>Ex</u>: R is an R-module. , submodules of R
are left (right) ideals.

$$End_R(R) = Hom_R(R,R) \xleftarrow{\sim} R \text{ as rings}$$

$$\varphi_r \longleftarrow r$$

$\varphi \in$

if $\varphi(1) = r$

$$\varphi(s) = s\varphi(1) = sr$$

$\varphi_r(s) = sr.$

<u>Ex</u> $R^n = \{(r_1, \ldots, r_n) \mid r_i \in R\}$ ptwise ops.

Ex R ring $I \trianglelefteq_{\ell} R \implies I$ left R-mod

left ideal — as is R/I.

Def If $N < M$ (N is an R-submod of M)

then the quotient Ab. gp M/N has the structure of an R-module via

$$r(m+N) = rm + N.$$

And have all standard isom. thms.

if $\varphi : M \longrightarrow N$ any R-mod hom,

$K = \ker\varphi = \{ m \in M \mid \varphi(m) = 0 \} < M$

and get an induced map $M/K \to N$

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & N \\ & M/K \arrow[ur] & \end{tikzcd}$$

where $M/K \to N$ is injective

so get an isom $M/K \cong \operatorname{im}\varphi$.

Def R is a ring, define R^{op} ring w/ same underlying Ab.gp but w/ mult.

$$r \cdot_{op} s = sr$$

Rem M is a right R module $\rightsquigarrow$ M can also be regarded as a left R^{op}-module.

$$r(sm) = (rs)m \qquad m(rs) = (mr)s$$

Def If R, S rings, an R-S bimodule is a Ab. gp M w/ left R-module & right S-module structures s.t. $(rm)s = r(ms)$.

Ex: V a right F-vector space $R = \mathrm{End}_F(V)$

$$\varphi(v\lambda) = \varphi(v)\lambda = (\varphi \cdot v)\lambda$$

V a $\mathrm{End}_F V$ - F bimodule.