More module definitions

If M is an R-module, $N_i < M$, $i \in I$

define $\sum_{i \in I} N_i$ to be $\{ \sum_{\text{finite sum}} n_i \mid n_i \in N_i \}$ (all but finitely many $= 0$)

if $A \subset M$ subset,

$$RA = \{ \sum_{\text{finite}} r_i a_i \mid r_i \in R, a_i \in A \} = \langle A \rangle$$

smallest submod of M containing A.

<u>Def</u> $R\{a\} = Ra$, if $M = Ra$ some a,

we say M is cyclic.

<u>Def</u> Given a collection of R-mods M_i $i \in I$

Define $\bigoplus_{i \in I} M_i$ to be set of finite formal sums

$$\sum_{i \in I} m_i$$

finitely many $\neq 0$ terms

"

$(m_i)_{i \in I}$ tuple w/ only finitely many $\neq 0$ terms

(external direct sum)

<u>universal property</u>: $\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i, N\right)$

$$\| $$

$$(f_i) \in \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$$

$$\bigoplus M_i \xrightarrow{\text{" }\oplus f_i\text{ "}} N$$

$$\sum_{i\in J} m_i \longmapsto \sum_{i\in J} f_i(m_i)$$

$J \subset I$ finite

<u>Def/Rem</u> $\prod_{i\in I} M_i$ has the structure of an R-module

$$r(m_i)_{i\in I} = (rm_i)_{i\in I}.$$

<u>Universal property</u>

$$\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) = \prod_{i\in I} \mathrm{Hom}_R(N, M_i)$$

$$\begin{bmatrix} N \longrightarrow \prod M_i \\ n \longmapsto (f_i(n))_{i\in I} \end{bmatrix} \longleftarrow (f_i)$$

Def If M an R-mod, $M_i < M$ $i \in I$ submods such that

- $\sum M_i = M$
- independent, i.e. $\sum_{i \in J} m_i = 0 \iff m_i = 0$ all i

then we say M an internal direct sum

$$\dot{\bigoplus_{i \in I}} M_i = M$$

(and we get $\bigoplus M_i \simeq M$)

Tensor Products

Def M in Mod-R (right R-mod)

N in R-Mod (left R-mod)

P an Abelian gp (in Ab)

A bilinear map $\varphi : M \times N \longrightarrow P$

is a map of sets such that

1) $\varphi(m_1+m_2, n) = \varphi(m_1,n)+\varphi(m_2,n)$

2) $\varphi(m, n_1+n_2) = \varphi(m,n_1)+\varphi(m,n_2)$

3) $\varphi(mr,n) = \varphi(m,rn)$

<u>Def</u> A $\otimes$ product $M\otimes_R N$ is an Abelian gp unique up to isom. w/ following universal property:

$$\mathrm{Bil}(M\times N,P) = \mathrm{Hom}_{Ab}(M\otimes_R N, P)$$

<u>Def</u> The $\otimes$ product of M,N is

$$M\otimes_R N = \frac{\langle M\times N\rangle}{R}$$

Free Abelian gp generated by the set $M\times N$

subgp gen. by all elmts of the form

$(m_1+m_2,n) - (m_1,n) - (m_2,n)$

$(m,n_1+n_2) - (m,n_1) - (m,n_2)$

$$(mr,n) - (m,rn)$$

$$\mathrm{Hom}_{Ab}(M \otimes_R N, P) = \{ f \in \mathrm{Hom}_{Ab}(\langle M \times N \rangle, P) \mid f(\mathcal{R}) = 0 \}$$

$$= \{ \tilde{f} \in \mathrm{Hom}_{Sets}(M \times N, P) \mid \tilde{f}(m_1+m_2, n) = \tilde{f}(m_1,n) + \tilde{f}(m_2,n) \text{ etc...} \}$$

$$= \mathrm{Bil}(M \times N, P)$$

$M \otimes_R N$ gen. by reps in $M \times N$

notation $m \otimes n \longleftarrow \overline{(m,n)}$

$\uparrow$

"simple tensors"

$$mr \otimes n = m \otimes rn \qquad (m_1+m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

<u>ex</u>: $R = F$ field V, W R-modules

recall: $\langle , \rangle : V \times W \longrightarrow P$ bilnr means

$$\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle \; . \; . \; . \; . \; .$$

Observe: image of $\langle , \rangle$ is closed under mult. by R.

$\Rightarrow$ image is contained in an R-mod.

$\lambda \langle v,w\rangle \equiv \langle \lambda v, w\rangle$ $\quad$ v,w

$\quad\quad\quad\quad\quad\quad\quad\;\; \shortparallel$ $\quad\quad\quad$ $V \;\; W$

$\langle v, \lambda w\rangle$ true if R comm.

if $\{e_i\}$ basis for V, $\{f_i\}$ basis for W

if we know $\langle e_i, f_j\rangle = \alpha_{ij} \in P$

then $\langle \sum a_i e_i, \sum b_j f_j\rangle$

$$= \sum_{ij} a_i b_j \langle e_i, f_j\rangle = \sum a_i b_j \alpha_{ij}$$

Conversely, given M, N, P, and elements $\alpha_{ij} \in P$

would like to define

$$\langle \sum a_i e_i, \sum b_j f_j\rangle = \sum a_i b_j \alpha_{ij}$$

this makes sense if P is an R-module

So, if R commuting

$$\mathrm{Bil}(M \times N, P) = M_{n,m}(P)$$

$M = R^n \qquad N = R^m$

$$x = [x_1, \dots, x_n] \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$$\langle x, y \rangle = x A y \qquad A = (\alpha_{ij})$$

A
"Gram matrix"

<u>Recall</u>: Symm. bilnr frm

$V \times V \xrightarrow{b} P$ vectr spaces over field / comm. ring R

$b(v,w) = b(w,v)$

$\Updownarrow$

$V \otimes_R V \xrightarrow{b} P$

$b(v \otimes w) = b(w \otimes v)$

<u>Recall</u> Skew-sym. bil. $\omega : V \times V \longrightarrow P$

$\omega(v,w) = -\omega(w,v)$

<u>Recall</u> Alternating form $a : V \times V \longrightarrow P$

$a(v,v) = 0$ all $v \in V$.

Alternating $\Rightarrow$ Skew symmetric

$$0 = a(v+w, v+w) = \overset{0}{a(v,v)} + \overset{0}{a(w,w)}$$

$+a(v,w)+a(w,v)$

Skew-Sym $\Rightarrow$ Alt. unless $2=0$

$\omega(v,v) = -\omega(v,v) \Rightarrow 2\omega(v,v)=0.$

<u>Ex</u> $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z}$

bilnr maps $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \xrightarrow{\varphi} P$

$$0 = \varphi(2a,b) = \varphi(a,2b) = \varphi(a,-b)$$
$$= -\varphi(a,b)$$

$$\varphi(2a,b) = \varphi(0,b) = \varphi(0,b)+\varphi(0,b)$$

$\varphi \equiv 0.$

$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0.$

<u>Ex</u>

$\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$

Pf: $\mathrm{Bil}(\mathbb{Z}^2 \times \mathbb{Q}, P) \rightleftarrows \mathrm{Hom}(\mathbb{Q}^2, P)$

$$\varphi \rightsquigarrow \left[\left(\tfrac{a}{b}, \tfrac{c}{d}\right) \mapsto \varphi\left((ad, bc), \tfrac{1}{bd}\right)\right]$$

$$(a,b)\otimes r \longrightarrow (ar, br)$$

$$(a,b)\otimes \frac{1}{c} \longleftrightarrow \left(\frac{a}{c}, \frac{b}{c}\right)$$