Tensor products

Examples

$$\mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$$

$$(a,b) \otimes p/q \longmapsto (ap/q,\ bp/q)$$

$$(ad, cb) \otimes \left(\frac{1}{bd}\right) \longleftarrow (a/b, c/d) = \left(\frac{ad}{bd}, \frac{cb}{bd}\right)$$

well-defined?

$$\lambda(a,b) \otimes p/q \longmapsto (\lambda ap/q,\ \lambda bp/q)$$

$$\| $$

$$(a,b) \otimes \lambda p/q$$

etc.

$$\big((a,b) + (a',b')\big) \otimes p/q \longmapsto$$

$$\|$$

$$\big((a,b) \otimes p/q\big) + (a',b') \otimes p/q$$

__More generally:__

If R, S rngs, $\varphi : R \to S$ rng hom.

can consider S as being in R-mod-R

and if M is a left R-module, $S \otimes_R M$

has the structure of a left S-module

via $s \cdot (s' \otimes m) = ss' \otimes m$

$s \sum s_i \otimes m_i = \sum s s_i \otimes m_i$

$$(s s')(s'' \otimes m) = s(s'(s'' \otimes m))$$

__Note:__ if N is a left S-module, $R \xrightarrow{\varphi} S$ gives

N a left R-module structure as well.

__Still more generally:__ R, S, T rngs

M in R-Mod-S

N in S-Mod-T then $M \otimes_S N$ in R-Mod-T

$r(m \otimes n) = rm \otimes n$

$(m \otimes n)t = m \otimes nt$

$(rm)s = r(ms)$

$m \otimes 1 \longleftarrow m$

$m \otimes r \longmapsto mr$

<u>Ex:</u> $M \otimes_R R \cong M$ as right R-modules

R-mod-R

<u>Ex:</u> $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}/n\mathbb{Q} = (0)$

$M \otimes_R R/I \simeq M/MI$

$m \otimes \bar{r} \longmapsto \overline{mr}$

$m \otimes \bar{1} \longleftarrow \bar{m}$

M in mod-R $\quad I \triangleleft R$ ideal

<u>Properties of $\otimes$</u>

given M, M' in S-mod-R

and N, N' in R-mod-T

$f : M \longrightarrow M'$ $\quad$ $g : N \longrightarrow N'$

get induced map

$$f \otimes g : M \otimes_R N \longrightarrow M' \otimes_R N'$$

$$(f \otimes g) \sum_i m_i \otimes n_i = \sum_i f(m_i) \otimes g(n_i)$$

Makes $\otimes_R$ into a bifunctor

$$\otimes_R : (S\text{-mod-}R) \times (R\text{-mod-}T) \to (S\text{-mod-}T)$$

canonical

$$\underset{U\text{-mod-}R}{M} \otimes_R (\underset{R\text{-mod-}S}{N} \otimes_S \underset{S\text{-mod-}T}{P}) \simeq (M \otimes_R N) \otimes_S P$$

$$m \otimes (n \otimes p) \longmapsto (m \otimes n) \otimes p$$

Universal trilinear thing $\quad M \otimes_R N \otimes_S P$

$$\mathrm{Hom}_{\substack{U\text{-}T \\ \text{bimodules}}}(M \otimes_R N \otimes_S P, Q) = \text{Trilinear maps } M \times N \times P \to Q$$

Makes sense to write (unambiguously)

$$M_0 \otimes_{R_1} M_1 \otimes_{R_2} M_2 \otimes \cdots \otimes_{R_{n-1}} M_{n-1} \in R_0\text{-mod-}R_n$$

in R_0-mod-R_1

in R_{n-1}-mod-R_n

<u>Distributivity</u>

in S-mod-T

$$M \otimes_R (N \oplus N') \simeq (M \otimes_R N) \oplus (M \otimes_R N')$$

M in S-mod-R $\quad$ N, N' in R-mod-T

<u>Pf</u>: $m \otimes (n, n') \longmapsto ((m \otimes n), (m \otimes n'))$

$m \otimes (n, 0) \longleftarrow (m \otimes n, 0)$

$m \otimes (0, n') \longleftrightarrow (0, m \otimes n')$

<u>Cor</u> $R \xrightarrow{\varphi} S$ ring hom,

$$S \otimes_R R^n \simeq S^n \qquad (S \otimes_R R = S)$$

Algebras

Def Let R be a commutative ring

An R-algebra is a ring A which is also an R-module such that $r(ab) = (ra)b = a(rb)$ all $a,b \in A$, $r \in R$

In this case, get a ring homomorphism

$$\begin{aligned} R &\longrightarrow A \\ r &\longmapsto r \cdot 1 \end{aligned}$$

Alt Def: An R algebra is a ring A, together with a ring hom. $R \to Z(A)$

$$Z(A) = \{a \in A \mid ab = ba \text{ all } b \in A\}$$

Ex: $M_n(F)$ or $M_n(R)$ R comm.

$$M_n(R) \cong R^{n^2}$$

$R[x]$ or $R[[x]]$

Proposition If A, B are R-algebras, $A \otimes_R B$ is an R-algebra via:

$$(a \otimes b)(c \otimes d) = ac \otimes bd \quad (\text{\& extend by linearity})$$

ex: $M_n(R) \otimes_R S \simeq M_n(S)$

$\varphi: R \to S$ $\qquad r \cdot s \equiv \varphi(r)s$

$(a_{ij}) \otimes s \longmapsto (a_{ij}s)$

$R[x] \otimes_R S \simeq S[x]$

$\left(\sum a_i x^i\right) \otimes s \longmapsto \sum a_i s x^i$

$R[[x]] \otimes_R S \not\simeq S[[x]]$

but works sometimes.

$A \otimes_R B$

$\left(\sum_i a_i \otimes b_i\right)\left(\sum_j c_j \otimes d_j\right)$

$" = " \sum_{i,j} (a_i \otimes b_i)(c_j \otimes d_j)$

$" = " \sum_{i,j} a_i c_j \otimes b_i d_j$

$$M_n(R) \otimes S \simeq M_n(S)$$

$$M_n(R) \otimes M_m(R) = M_n(M_m(R))$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$M_m(M_n(R)) = M_{nm}(R)$$