If A & B are commutative R-algebras then $A \otimes_R B$ is the "categorical sum" of A & B

i.e. $$\mathrm{Hom}_{\substack{R\text{-alg}\\ \text{com}}}(A \otimes_R B, C) = \mathrm{Hom}_{\substack{R\text{-alg}\\ \text{com}}}(A, C) \times \mathrm{Hom}_{\substack{R\text{-alg}\\ \text{com}}}(B, C)$$

More generally, if A, B not nesc. commutative

$$\mathrm{Hom}_{R\text{-alg}}(A \otimes_R B, C) = \{ f : A \to C,\ g : B \to C \mid f(a)g(b) = g(b)f(a),\ \text{all } a \in A,\ b \in B \}$$

Pf sketch:

$$[\varphi : A \otimes_R B \to C] \longmapsto \left[\begin{array}{l} A \to A \otimes_R B \xrightarrow{\varphi} C, \\ a \mapsto a \otimes 1 \\ B \to A \otimes B \xrightarrow{\varphi} C \\ b \mapsto 1 \otimes b \end{array} \right]$$

$$\left[\begin{array}{l} A \otimes B \longrightarrow C \\ a \otimes b \longmapsto f(a)g(b) \end{array} \right] \longleftarrow (f, g)$$

$\square$

Adjointness of $\otimes$ & restriction of scalars

$R \xrightarrow{\psi} S$ N an R-module

L an S-module

<u>Def</u> ψL = L thought of as an R-module via ψ.

$$\boxed{\mathrm{Hom}_R(N, \psi L) = \mathrm{Hom}_S(S \otimes_R N, L)}$$

$$\mathrm{Hom}_R(N, \psi L) \leftarrow \mathrm{Hom}_S(S \otimes N, L)$$

$$\begin{tikzcd} N \arrow[r] \arrow[rr, bend left, dashed] & S \otimes_R N \arrow[r] & L \end{tikzcd}$$

$$n \longmapsto 1 \otimes n$$

<u>Thm (in book)</u>:

given $\varphi : N \to L$ $\exists!$ $\Phi : S \otimes_R N \to L$

such that diagram commutes:

$$\begin{tikzcd} N \arrow[r, "n \mapsto 1 \otimes n"] \arrow[rr, "\varphi"'] & S \otimes_R N \arrow[r, "\Phi"] & L \end{tikzcd}$$

$\otimes$'s over commutative rings

$$\left(\begin{array}{cc} R\text{-mod-}S & S\text{-mod-}T \\ M & N \\ & \\ \multicolumn{2}{c}{M \otimes_S N} \\ \multicolumn{2}{c}{\text{in } R\text{-mod-}T} \end{array} \right)$$

if R is commutative, M an R-mod, can define an R-R bimod structure on M by $m \cdot r \equiv rm \qquad r \cdot m = rm$

$$r(mr') = r r' m = r' r m$$
$$= (rm) r'$$

M, N R-mods, can consider both as R-R bimods

& $\otimes_R$ gives another R-R-bimod
" R-mod

$$r(m \otimes n) \equiv rm \otimes n = mr \otimes n = m \otimes rn$$
$$= m \otimes nr$$

Digression

Let F be a field of characteristic $\neq 2$.

Let V/F be a vector space and consider

$$Bil(V) = \{ V \times V \xrightarrow{b} F \mid b \text{ is bilinear} \}$$

$$\parallel$$

$$(V \otimes_F V)^* \ni \varphi \quad \text{get} \quad \begin{aligned} V \times V &\to F \\ (v,w) &\mapsto \varphi(v \otimes w) \end{aligned}$$

Claim: if $b \in Bil(V)$, $\exists!\, b', b''$ such that b' is skew-symmetric, b'' is symmetric & $b = b' + b''$

Let $\sigma : Bil(V) \to Bil(V)$ defined by

$$\sigma b(v,w) \equiv b(w,v)$$

Define a hom. of rings (F-algebras) $F[x] \to End_F(Bil(V))$

$$x \mapsto \sigma$$

note: $x^2 - 1 \mapsto 0$ so get

$$R = F[x]/_{x^2-1} \to End_F(Bil(V))$$

$\Rightarrow$ Bil(V) has the structure of an R-module.

$$xb \equiv \sigma b$$
$$\left(\sum a_i x^i\right) b = \sum a_i \sigma^i(b)$$

$$R = F[x]/x^2-1 = F[x]/(x-1)(x+1)$$
$$= F[x]/x-1 \times F[x]/x+1$$
$$= F \times F$$

$\frac{1}{2}(x+1)$ " $e \leftrightarrow (1,0)$ $\quad f \leftrightarrow (0,1)$ " $-\frac{1}{2}(x-1)$

$1 = e+f$

$$e^2 = e \quad f^2 = f \quad ef = fe = 0$$

$$xe = (1,-1)(1,0) = (1,0) = e$$
$$xf = (1,-1)(0,1) = (0,-1) = -f$$

in particular, given $b = 1b = (e+f)b = eb + fb$

$\sigma(eb) = x(eb) = (xe)b = eb$ $\quad$ eb symm

$\sigma(fb) = x(fb) = (xf)b = -fb$ $\quad$ fb skew

$b = eb + fb$

uniqueness: note

$b = b' + b'' = c' + c''$ $\quad$ b', c' sym

b'', c'' skew

$\underbrace{b' - c'}_{\text{sym}} = \underbrace{c'' - b''}_{\text{skew}}$

if d is skew & symm

$\Rightarrow \sigma d = d \Rightarrow 2d = 0$

$\sigma d = -d$

$d = 0$.

Vector Spaces (Ch 11)

Def A vector space = a module over a field
(skew-field)
" division ring.

Def If V a vspace$/F$
and $S \subset V$ is a subset, we say S is
independent if $\sum_{i=1}^{n} a_i s_i = 0$, $s_i \in S$ distinct

$\iff a_i = 0$ all i.

Def S spans V is all $v \in V$ can be written as

$\sum a_i s_i = v \quad a_i \in F, s_i \in S.$

Def S is a basis if it is an independent
spanning set.

Prop If S is a spanning set such that no
proper subset spans then S is a basis.
& conversely.

Pf: Basis $\Rightarrow$ min'l spanning

if $S \setminus \{s\}$ spans $\Rightarrow$ $s = \sum a_i s_i \quad s_i \neq s$

$\Rightarrow \sum a_i s_i - 1 \cdot s = 0$

~~not~~ indep.

$\Leftarrow$

indep since if

$$\sum a_i s_i = 0 \ , \ a_1 \neq 0$$

$$\Rightarrow s_1 = -a_1^{-1} \sum_{i>1} a_i s_i$$

$$s_1 = \sum_{i>1} (-a_1^{-1} a_i) s_i$$

not minimal.

Cor any spanning set contains a basis

Prove it yourself, or will discuss later.