Read:

bases exist, have same size,

$T: V \to W$ $\qquad$ $\dim \operatorname{im}(T) + \dim \ker(T) = \dim V$

max'l indep
$\|$
min'l spanning

if $\dim V = \dim W < \infty$

TFAE: T iso

T inj

T surj

Vector space $\Rightarrow$ free modules

arbitrary indep sets extend to bases

V a right D-space $\qquad$ D div. ring.

$V^* = \operatorname{Hom}_{r.D\text{-space}}(V, D)$ is a left D-space.

$$\boxed{\text{via: } (\partial f)(v) \equiv \partial(f(v)) \qquad \partial \in D}$$

$$(\partial f)(va) = \partial(f(va)) = \partial(f(v)a)$$

$a \in D$ $\qquad$ $\|?$ $\qquad$ $= (\partial f(v))a$

$$((\partial f)(v))a \qquad = ((\partial f)(v))a$$

V is a left D space $\quad V^*$ right D space.

$$(fd)(v) \equiv (f(v))d \quad \ldots$$

Matrices

$V \qquad W \qquad$ right D-spaces

$\{v_i\} \qquad \{w_j\} \qquad$ bases

$\varphi : V \longrightarrow W \qquad$ lin. trans. (i.e. D-map)

$$\varphi(v_i) = \sum_j w_j \varphi_{ji} \qquad \varphi_{ji} \in D$$

$$\varphi\left(\sum_i v_i a_i\right) = \sum_i \varphi(v_i) a_i$$

$$= \sum_{i,j} w_j \varphi_{ji} a_i$$

$$\begin{bmatrix} \varphi_{ji} \end{bmatrix} \begin{bmatrix} a_i \end{bmatrix}$$

Let's switch to fields (for comfort)

$V/F \qquad F \to E$ fields

$E \otimes_F V$ is an E-vector space.

if $\{v_i\}$ basis for V then $\{1 \otimes v_i\}$ basis for $E \otimes_F V$

$$V \simeq F^n = \bigoplus_{i=1}^{n} F$$

$$E \otimes_F V \simeq E \otimes \left(\bigoplus_i F\right) \simeq \bigoplus_i E \otimes_F F$$

$$= \bigoplus_i E$$

Similarly, if V has basis $\{v_i\}$

W - - - $\{w_j\}$

$V \otimes_F W$ has a basis $\{v_i \otimes w_j\}$

$$\left(\bigoplus F v_i\right) \otimes \left(\bigoplus F w_j\right) \simeq \bigoplus_{i,j} \left(F v_i \otimes F w_j\right)$$

$V \otimes_F W$ is an F space of dim $= (\dim V)(\dim W)$

$\otimes$ of matrices aka Kronecker product

$f : V \longrightarrow V'$ $\qquad$ $g : W \longrightarrow W'$

$$f \otimes g : V \otimes_F W \longrightarrow V' \otimes_F W'$$

$$f \otimes g\,(v \otimes w) = f(v) \otimes g(w)$$

$f \leftrightarrow (f_{ij})$ $\qquad$ $g \leftrightarrow (g_{kl})$

$f(v_i) = \sum f_{ij} v'_j$ $\qquad$ $g(w_k) = \sum g_{kl} w'_l$

$$(f \otimes g)(v_i \otimes w_k) = \left(\sum_j f_{ij} v'_j\right) \otimes \left(\sum_l g_{kl} w'_l\right)$$

$$= \sum_{jl} f_{ij} g_{kl} (v'_j \otimes w'_l)$$

A matrix for f

B matrix for g $\qquad$ $f \otimes g$ has matrix called $A \otimes B$

Duals & double duals

$V^* = \mathrm{Hom}(V, F)$, if V has basis $\{v_i\}$

then get dual "basis" $f_i \in V^*$ via

$f_i\left(\sum a_j v_j\right) = a_i$ can see they are independent,

span if V is f.dim'l.

$$\left(\bigoplus V_i\right)^* = \mathrm{Hom}\left(\bigoplus V_i, F\right)$$
$$= \prod \mathrm{Hom}(V_i, F)$$
$$= \prod (V_i^*) \qquad V_i \cong F$$

$$\left(\bigoplus_i F\right) \longrightarrow \left(\prod_i F\right)$$
$$V \longrightarrow V^*$$ given by dual basis construction

If finite dim'l, everything is free, this is an iso.

$$V \longrightarrow V^{**}$$

$$v \longmapsto (f \longmapsto f(v))$$

an iso iff f.dim'l.

Aside: M module over comm ring R

$$M^{**} \leftarrow M$$

$\simeq$ is called "reflexive"

V a vector space, the tensor algebra $T(V)$

"free algebra over F generated by V"

natural isom of bifunctors $(\text{Vspace})^{op} \times (F\text{-alg}) \to \text{Sets}$

$$\mathrm{Hom}_{F\text{vspce}}(V, A) \simeq \mathrm{Hom}_{F\text{Alg}}(T(V), A)$$

defines it up to unique isomorphism.

i.e. $T(V)$ is an F-alg. w/ vspace map $V \to T(V)$

such that $\forall$ F-alg A, vspace maps $V \to A$

$\exists!\ T(V) \to A$ such that

$$\begin{tikzcd} T(V) \arrow[rr] & & A \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$ commutes.

If B is another F-alg, together with a map $V \to B$ such that $\forall A$, $V \to A$

$\exists!\ B \to A$ s.t.
$$\begin{tikzcd} B \arrow[rr] & & A \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$ commutes.

Then $\exists!$ isom $T(V) \to B$ such that

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & T(V) \arrow[d, "\wr"] \\ & B \end{tikzcd}$$ commutes.

<u>Def</u> $T(V) = F \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$

$$\bigoplus_{i=0}^{\infty} \left(\bigotimes^{i} V \right)$$

$$(a_1 \otimes \cdots \otimes a_n)(b_1 \otimes \cdots \otimes b_m) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m$$

$V \xrightarrow{\varphi} A \qquad T(V) \longrightarrow A$

$v_1 \otimes \cdots \otimes v_m \longmapsto \varphi(v_1)\varphi(v_2)\cdots\varphi(v_n)$

ex: $V = 1$ dim'l $= \langle x \rangle$

$V \otimes V$ 1-dim basis $x \otimes x = x^2$

$\underbrace{V \otimes \cdots V}_{k\text{-times}} \qquad x \otimes \cdots \otimes x = x^k$

$x^i x^j = x^{i+j}$

$T(V) = F[x]$.

$V = \langle x, y \rangle$

$V \otimes V = \langle x \otimes x, x \otimes y, y \otimes x, y \otimes y \rangle$

$V \otimes V \otimes V = \langle \qquad \rangle$ 8 dim'l

$T(V)$ is a graded ring

Def A graded ring is a ring S together with a decomposition $S = S_0 \oplus S_1 \oplus \cdots$ into subgroups under addition such that $S_i S_j \subset S_{i+j}$, $1 \in S_0$

Note: S_0 always a subring.

Graded algebra $\Longleftrightarrow$ S_i's are subspaces.

Def The elements of S_i's are called homogeneous

Def $I \triangleleft S$ is called a graded ideal if $I = \bigoplus_{i=0}^{\infty} (I \cap S_i)$

Prop $I \triangleleft S$ graded ideal $\Longleftrightarrow$ I can be generated by homogeneous elements.

Pf: $\Rightarrow$ by def I is gen as an Ab. gp by hom elmts.

$\Leftarrow$ suppose $I = \langle x_i \rangle_{i \in \Lambda}$

hom x_i deg n_i

$x \in I \Rightarrow x = \sum a_i x_i \quad a_i \in S$

$S = \bigoplus S_i \Rightarrow a_i = \sum b_{ij}, \; b_{ij} \in S_j$

$x = \sum_i \sum_j \underbrace{b_{ij} x_i}_{\text{hom elmt of } I}$ $\square$

<u>Prop</u> S graded, $I \triangleleft S$ graded $\Rightarrow S/I$ graded

via $(S/I)_i \cong S_i/I_i \qquad I_i = S_i \cap I$

<u>Pf:</u>

$$S = S_0 \oplus S_1 \oplus \cdots \longrightarrow \underbrace{S_0/I_0 \oplus S_1/I_1 \oplus \cdots}_{\text{graded ring}}$$

$\bar{s}_i \cdot \bar{s}_j = \overline{s_i s_j}$ in S_{i+j}/I_{i+j}

$S_i/I_i \quad S_j/I_j$

$\ker = I_0 \oplus I_1 \oplus \cdots$

$= I$ $\square$