

Recap

n -ary operations on a set

$$\omega: S^n \longrightarrow S$$

Given Ω a set and a function $\Omega \rightarrow \mathbb{N}$
"arity"

Define an Ω -algebra to be a set A

together with maps $\omega: A^n \longrightarrow A$ $\omega \in \Omega$
w/ arity n .

Def If A, B are Ω -algs, $f: A \rightarrow B$

is an Ω -alg hom if $f(\omega(a_1, \dots, a_n)) =$
 $\omega(f(a_1), \dots, f(a_n))$

we say a subset $A \subset B$, both Ω algebras is
an Ω -subalg. if the inclusion $A \hookrightarrow B$
is an Ω -alg hom.

(\Leftrightarrow A closed under operators)

Def Given $\Omega \rightarrow \mathbb{Z}_{>0}$ and a collection of identities, we can consider the class of Ω -algebras satisfy these identities. Such a class is called a variety.

Ex: Class of groups, class of rings.

Groupoids

Ex: let S_1, \dots, S_m be finite sets

$$\mathcal{G} = \{ f: S_i \rightarrow S_j \mid f \text{ bijective} \}$$

Notation:

$$\text{given } f: S_i \rightarrow S_j$$

$$t(f) = S_j \quad s(f) = S_i$$

Def $f \circ g$ whenever $s(f) = t(g)$

$$s(f \circ g) = s(g) \quad t(f \circ g) = t(f)$$

$$e_{s_i}: S_i \rightarrow S_i \text{ identity } t(e_{s_i}) = s(e_{s_i}) = s_i$$

$$f e_{s_i} = f \quad e_{t(f)} f = f$$

$$\forall f, \exists f^{-1} \quad s(f) = t(f^{-1}) \quad s(f^{-1}) = t(f)$$

$$f^{-1} f = e_{s(f)} \quad f f^{-1} = e_{t(f)}$$

$$\xi (f \cdot g) \cdot h = f \cdot (g \cdot h) \text{ when defined.}$$

Def A groupoid \mathcal{G} is a pair of sets G_1, G_0 together with $s, t: G_1 \rightarrow G_0$

"arrows" \nearrow
"vertices" \nearrow

and a composition law

$$\{(g, g') \mid s(g) = t(g')\} \longrightarrow G_1$$

$$g, g' \longmapsto g \cdot g'$$

$$s(g \cdot g') = s(g') \\ t(g \cdot g') = t(g)$$

and identity elmts

$$e: G_0 \longrightarrow G_1$$

$$s(e_v) = t(e_v) = v$$

$$v \longmapsto e_v$$

inverses $\nu: G_1 \rightarrow G_0$ $s(\nu(g)) = t(g)$
 $g \rightarrow g^{-1}$ $t(\nu(g)) = s(g)$

such that

• associativity $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ when defined

....

Note: if $|G_0| = 1$ we are really talking about groups.

Examples:

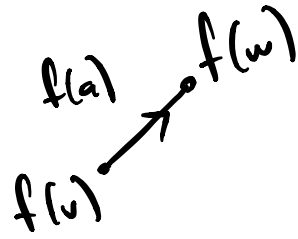
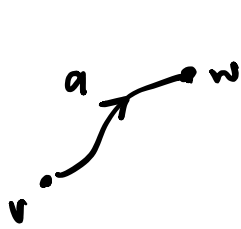
Choose a collection of sets, groups, rings, vector spaces, fields, top spaces, isoms between them.

Def Given groupoids $\mathcal{G} = (G_1, G_0)$
 $\mathcal{H} = (H_1, H_0)$

a homomorphism $f: \mathcal{G} \rightarrow \mathcal{H}$

is a pair of maps $f_1: G_1 \rightarrow H_1$ $(f = f_1 = f_0$
 $f_0: G_0 \rightarrow H_0$ if lazy)

such that:

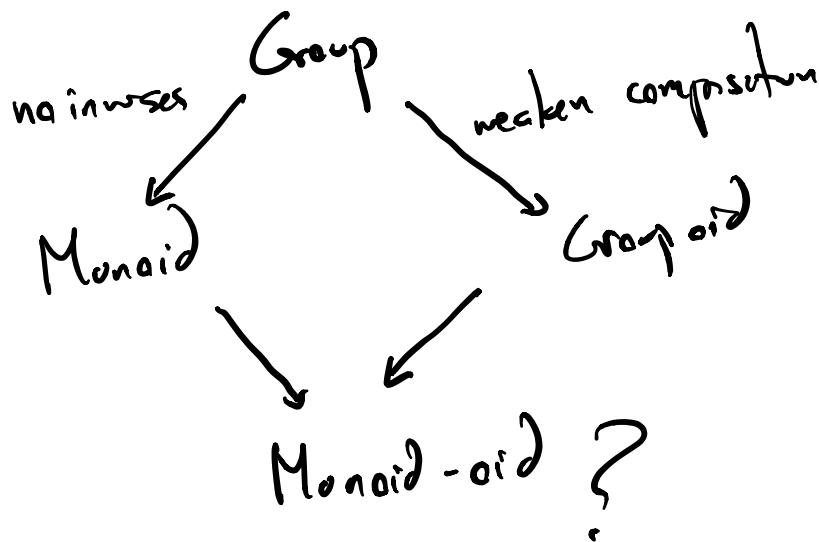


$$s(f_1(a)) = f_0(s(a)) \text{ etc.}$$

$$f_1(e_v) = e_{f_0(w)} \quad f_1(ab) = f_1(a) f_1(b).$$

Ex: "forgetful map" Groups \rightarrow Sets.

Ex: "basis" Set \rightarrow v.space / $F = \mathbb{C}$
 $S \rightarrow$ v.space w/ basis S
 $\{ \sum a_s \cdot s \mid a_s \in \mathbb{C} \}$



Def A monoid-oid consists of

C_0 objects C_1 arrows

$s, t: C_1 \rightarrow C_0$ $e: C_0 \rightarrow C_1$

$s(e_v) = t(e_v) = v$

comp law

$\{ (a, b) \mid s(a) = t(b) \} \rightarrow C_1$

l.s.t.c. Associativity
Unit/identity arrows

Def A category is a monoid-oid.

Ex Finite dim'l vector spaces.
↳ linear transformations between them

$C_0 = \text{f.d. v. spaces}$

$C_1 = \{ T: V \rightarrow W \}$

FDVect

Ex: "Matrices" $C_1 = \{ M_{n,m}(\mathbb{R}) \mid n, m \in \mathbb{Z}_{\geq 1} \}$

$C_0 = \mathbb{Z}_{\geq 1}$

Mat

$$\text{equiv. } \mathcal{C}_0 = \{ \mathbb{R}, \mathbb{R}^2, \mathbb{R}^3, \dots \}$$

Def homomorphism of categories same def as for groupoids, called "Functors"

Obsene have a functor

$$\underline{\text{Mat}} \longrightarrow \underline{\text{FDVect}}$$

$$n \longmapsto \mathbb{R}^n \quad \text{object}$$

$$[a_{ij}] \longmapsto \text{corresp. lin trans.}$$

Note: A group \iff a category w/ one object
! all arrows invertible.

Actual Course, Part 1 (finite) groups.

Fundamental question: How are groups put together from smaller pieces?

Given G , - what are its subgroups?
- how do they fit together?

Def A subgroup $N < G$ is normal if

$$gNg^{-1} = N \quad N \triangleleft G$$

Normal subgroups are the same as kernels of homomorphisms

Given $N \triangleleft G$, can consider

$$G \twoheadrightarrow G/N \text{ w/ kernel } N$$

Imagine G is made up of $G/N \wr N$

Jordan-Hölder Program

- 1: Classify the simple groups (no nontr. normal subgps)
- 2: Classify all ways of putting them together (given $G/N \cong N$, what are possible G 's?)

Some special kinds of subgps

Given $H < G$, $|H| \mid |G|$ (Lagrange)
on the other hand, if $m \mid |G|$ when $\exists H < G$
w/ $|H| = m$?

Def If $m \mid |G|$, $(m, |G|/m) = 1$
and $H < G$ w/ $|H| = m$, we say that
 H is a Hall subgroup of G

(turns out these always exist if G is solvable)

Def if $m \mid |G|$ $(m, |G|/m) = 1$, $m = p^n$,
 $H < G$ w/ $|H| = m$, we say that H is
a p -Sylow subgroup.

These always exist.
will be a key tool.

Another perspective via automorphisms

Observe, given a group G , $g \in G$

Get a homomorphism
(automorphism)

$$\text{inn}_g: G \rightarrow G$$
$$h \mapsto ghg^{-1}$$

$$h'h \mapsto gh'hg^{-1} = g'h'g^{-1}ghg^{-1}$$
$$= \text{inn}_g(h') \text{inn}_g(h)$$

Normal subgroups are exactly those

$$\text{for all } g \in G, N < G \text{ s.t. } \text{inn}_g(N) = N$$

Can often find subgroups that are fixed by any automorphism.

ex1 $Z(G) = \{g \in G \mid gh = hg \text{ all } h \in G\}$

$$Z(G) < G$$

$Z(G)$ is preserved by any automorphism.

\Rightarrow preserved by conj \Rightarrow normal

$\{g \in G \mid g \text{ commutes w/ all elmts of order } 3\}$
is also preserved by any aut \Rightarrow normal.

Def A subgroup $H < G$ is called characteristic
($H \text{ char } G$) if \forall automorphisms $\varphi: G \rightarrow G$
we have $\varphi(H) = H$.

Ex1 $[G, G] = \langle \{ghg^{-1}h^{-1} \mid g, h \in G\} \rangle$

$$[G, G] \text{ char } G$$

$$[S, T] = \langle \{ sts^{-1}t^{-1} \mid s \in S, t \in T \} \rangle$$

$$S, T \subset G$$

$$[[G, [G, G]], [G, G]] \text{ char } G$$

Z 's and $[,]$'s.

Lem: if $H, K \text{ char } G \Rightarrow [H, K] \text{ char } G$