Recap

n-ary operations on a set

$$\omega : S^n \longrightarrow S$$

Given Ω a set and a function $\Omega \to \mathbb{Z}_{\geq 0}$
"arity"

<u>Define</u> an Ω-algebra to be a set A together with maps $\omega : A^n \longrightarrow A$, $\omega \in \Omega$ w/ arity n.

<u>Def</u> If A, B are Ω-alg s, $f : A \to B$ is an Ω-alg hom if $f(\omega(a_1, \ldots, a_n)) = \omega(f(a_1), \ldots, f(a_n))$

we say a subset $A \subset B$, both Ω algebras is an Ω-subalg. if the inclusion $A \hookrightarrow B$ is an Ω-alg hom.

($\Leftrightarrow$ A closed under operations)

Def Given $\Omega \to \mathbb{Z}_{\geq 0}$ and a collection of identities, we can consider the class of Ω-algebras satisfy these identities.

Such a class is called a variety.

Ex: Class of groups, class of rings.

Groupoids

Ex: let $S_1, \ldots, S_m$ be finite sets

$$\mathcal{G} = \{ f : S_i \to S_j \mid f \text{ bijective} \}$$

Notation:

given $f : S_i \to S_j$

$$t(f) = S_j \quad s(f) = S_i$$

Define $f \circ g$ whenever $s(f) = t(g)$

$$s(f \circ g) = s(g) \quad t(f \circ g) = t(f)$$

$e_{s_i} : S_i \to S_i$ identity $t(e_{s_i}) = s(e_{s_i}) = S_i$

$$f e_{s_i} = f \qquad e_j f = f$$

$\forall f,\ \exists f^{-1}$ $\quad s(f) = t(f^{-1}) \quad s(f^{-1}) = t(f)$

$$f^{-1} f = e_{s(f)} \qquad f f^{-1} = e_{t(f)}$$

& $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ when defined.

__Def__ A groupoid $\mathcal{G}$ is a pair of sets G_1, G_0

"arrows" "vertices"

together with

$$s, t : G_1 \to G_0$$

and a composition law

$$\{(g, g') \mid s(g) = t(g')\} \longrightarrow G_1$$

$$g, g' \longmapsto g \cdot g'$$

$s(gg') = s(g')$

$t(gg') = t(g)$

and identity elmts

$$e : G_0 \longrightarrow G_1 \qquad s(e_v) = t(e_v) = v$$

$$v \longmapsto e_v$$

inverses $\quad \iota : G_1 \longrightarrow G_1 \quad s(\iota(g)) = t(g)$

$\qquad\qquad g \longmapsto g^{-1} \quad t(\iota(g)) = s(g)$

such that

- associativity $\quad g\cdot(h\cdot k) = (g\cdot h)\cdot k$ when defined
- ....

<u>Note</u>: if $|G_0| = 1$ we are really talking about groups.

<u>Examples</u>:

Choose a collection of sets, groups, rings, v.spaces, fields, top spaces, isoms between them.

<u>Def</u> Given groupoids $\mathcal{G} = (G_1, G_0)$
$\mathcal{H} = (H_1, H_0)$

a homomorphism $f : \mathcal{G} \longrightarrow \mathcal{H}$

is a pair of maps $f_1 : G_1 \longrightarrow H_1$ $\quad$ ($f = f_1 = f_0$ if lazy)

$f_0 : G_0 \longrightarrow H_0$

such that:

$$v \xrightarrow{a} w \qquad\qquad f(v) \xrightarrow{f(a)} f(w)$$

$$s(f_1(a)) = f_0(s(a)) \quad \text{etc.}$$

$$f_1(e_v) = e_{f_0(v)} \qquad\qquad f_1(ab) = f_1(a)\,f_1(b).$$

Ex: "forgetful map" $\quad$ Groups $\longrightarrow$ Sets.

Ex: "basis" $\quad$ Set $\longrightarrow$ v.space/$F = \mathbb{C}$

$S \longmapsto$ v.space w/ basis S

$\{ \sum a_s \cdot s \mid a_s \in \mathbb{C} \}$

$$\begin{tikzcd}
 & \text{Group} \arrow[dl, "\text{no inverses}"'] \arrow[dr, "\text{weaken composition}"] & \\
\text{Monoid} \arrow[dr] & & \text{Groupoid} \arrow[dl] \\
 & \text{Monoid-oid ?} &
\end{tikzcd}$$

<u>Def</u> A monoid-oid consists of

C_0 objects $\quad C_1$ arrows

$$s,t : C_1 \to C_0 \qquad e : C_0 \to C_1$$

$$s(e_v) = t(e_v) = v$$

comp law

$$\{(a,b) \mid s(a) = t(b)\} \longrightarrow C_1$$

$\boxed{\text{s.t.}}$ Associativity

Unit/identity arrows

<u>Def</u> A category is a monoid-oid.

<u>Ex:</u> Finite dim'l vectorspaces.
& linear transformations between them

<u>FDVect</u>

C_0 = f.d. v.spaces

$C_1 = \{T : V \to W\}$

<u>Ex</u>: "Matrices" $\quad C_1 = \{M_{n,m}(\mathbb{R}) \mid n,m \in \mathbb{Z}_{\geq 1}\}$

$C_0 = \mathbb{Z}_{\geq 1}$

<u>Mat</u>

equiv. $\mathcal{C}_0 = \{\mathbb{C}, \mathbb{C}^2, \mathbb{C}^3, \dots\}$

Def homomorphism of categories same def as for groupoids, called "Functors"

Observe have a functor

$$\underline{\mathrm{Mat}} \longrightarrow \underline{\mathrm{FDVect}}$$

$$n \longmapsto \mathbb{C}^n \qquad \text{object}$$

$$[a_{ij}] \longmapsto \text{corresp. linr trans.}$$

Note: A group $\longleftrightarrow$ a category w/ one object & all arrows invertible.

Actual Course, Part 1 (finite) groups.

Fundamental question: How are groups put together from smaller pieces?

Given G, - what are its subgroups?
- how do they fit together?

Def A subgroup $N < G$ is normal if

$$gNg^{-1} = N \qquad N \triangleleft G$$

Normal subgroups are the same as kernels of homomorphisms

Given $N \triangleleft G$, can consider

$$G \twoheadrightarrow G/N \text{ w/ kernel } N$$

Imagine G is made up of G/N & N

Jordan-Hölder Program

1: Classify the simple groups (no nontriv. normal subgps)

2: Classify all ways of putting them together (given G/N & N, what are possible G's?)

Some special kinds of subgps

Given $H < G$, $|H| \mid |G|$ (Lagrange)

on the other hand, if $m \mid |G|$ when $\exists H < G$ w/ $|H| = m$?

<u>Def</u> If $m \mid |G|$, $(m, |G|/m) = 1$ and $H < G$ w/ $|H| = m$, we say that H is a <u>Hall subgroup</u> of G

(turns out these always exist if G is solvable)

__Def__ if $m \mid |G|$ $(m, |G|/m) = 1$, $m = p^n$,

$H < G$ w/ $|H| = m$, we say that H is a p Sylow subgroup.

These always exist.
will be a key tool.

__Another perspective__ via automorphisms

Observe, given a group G, $g \in G$

Get a homomorphism
(automorphism)

$$\mathrm{inn}_g : G \longrightarrow G$$
$$h \longmapsto ghg^{-1}$$

$$h'h \longmapsto gh'hg^{-1} = gh'g^{-1}ghg^{-1}$$
$$= \mathrm{inn}_g(h')\,\mathrm{inn}_g(h)$$

Normal subgroups are exactly those
for all $g \in G$, $N < G$ s.t. $\mathrm{inn}_g(N) = N$

Can often find subgroups that are fixed by <u>every</u> automorphism.

<u>ex:</u> $Z(G) = \{ g \in G \mid gh = hg \text{ all } h \in G \}$

$$Z(G) < G$$

$Z(G)$ is preserved by every automorphism.

$\Rightarrow$ preserved by conj $\Rightarrow$ normal

$\{ g \in G \mid g \text{ commutes w/ all elmts of order } 3 \}$

is also preserved by every aut $\Rightarrow$ normal.

<u>Def</u> A subgroup $H < G$ is called <u>characteristic</u> (H char G) if $\forall$ automorphisms $\varphi : G \to G$ we have $\varphi(H) = H$.

<u>Ex:</u> $[G, G] = \langle \{ g h g^{-1} h^{-1} \mid g, h \in G \} \rangle$

$[G, G]$ char G

$[S,T] = \langle \{ sts^{-1}t^{-1} \mid s \in S, t \in T \} \rangle$

$S, T \subset G$

$[[G,[G,G]],[G,G]]$ char G

Z's and $[\ ,\ ]$'s.

<u>**Lem:**</u> if H, K char $G \Rightarrow [H,K]$ char G