Tensor products / tensor algebras

$T(V)$ constructed — universal props

$T(V) \to A$ alg map
$\updownarrow$
$V \to A$ v.s. map

— explicitly via tensor product.

$V, \quad V \otimes V, \quad V \otimes V \otimes V$

Will define Exterior & Symmetric products

$\Lambda^i(V) \quad S^i(V)$ as components in a "universal" algebra as in $T(V)$

$\wr \quad \wr$

$V \otimes \cdots \otimes V$ i times

Today, we'll shift to modules over a comm. ring.

V is a free R-module, w/ basis x, y

$T(V)_0 = R \qquad T(V)_1 = V = Rx \oplus Ry$

$$T(V)_2 = V \otimes_R V = \underset{x \otimes x}{Rx^2} \oplus \underset{x \otimes y}{Rxy} \oplus \underset{y \otimes x}{Ryx} \oplus \underset{y \otimes y}{Ry^2}$$

$$T(V)_3 = Rx^3 \oplus Rx^2y \oplus Rxyx \oplus Ryx^2 \oplus Rxy^2 \oplus Ryxy \oplus Ry^2x \oplus Ry^3$$

$T(V)_i$ free of rank 2^i

Symmetric Algebra (V an R-module)

$S(V)$ is a comm. R-algebra, with an R-module inclusion $V \hookrightarrow S(V)$, with the univ. prop:

given any R-module map $V \to A$, A a comm. R-algebra, $\exists!$ extension $S(V) \to A$ of comm. R-algebras.

$$\mathrm{Hom}_{R\text{-mod}}(V, A) = \mathrm{Hom}_{\text{Comm-}R\text{-alg}}(S(V), A)$$

<u>In fact:</u> $S(V) = T(V)/C(V)$ $\quad C(V) \triangleleft T(V)$

$C(V)$ = ideal gen. by $v \otimes w - w \otimes v$ in $T(V)$

Ex: $V = Rx \oplus Ry$

$S(V)_0 = R$ $\qquad$ $S(V)_1 = V = Rx \oplus Ry$

$S(V)_2 = Rx^2 \oplus Rxy \oplus Ry^2$ $\qquad$ $xy = yx$

$S(V)_3 = Rx^3 \oplus Rx^2y \oplus Rxy^2 \oplus Ry^3$

$\operatorname{rk} S(V)_i = i+1$

$S(V) = R[x,y]$

In general, if V = free w/ basis $x_1, \dots, x_n$

$S(V) = R[x_1, \dots, x_n]$

$\operatorname{rk}(S(V)_i) = \binom{n+i-1}{n-1} = \binom{n+i-1}{i}$

$x_1 x_1 | x_2 x_2 x_2 | | x_4 | x_5 x_5$ $\qquad$ i things, $n-1$ separators

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Exterior Algebra

universal algebra containing V w/ $v^2 = 0$ all $v \in V$.

$\Lambda(V)$ an assoc. alg. containing V such that any R-module map $V \xrightarrow{\varphi} A$ to an assoc alg A satisfying $\varphi(v)^2 = 0$ has a unique extension to $\Lambda(V) \to A$.

<u>Def</u> $\Lambda(V) = T(V)/A(V)$ $\quad A(V) \trianglelefteq T(V)$

where $A(V)$ = ideal gen by elements of the form $v \otimes v$.

<u>note:</u> if 2 does not annihilate any elmt of M, then get same def by saying

$A(V)$ gen by $\quad v\otimes w + w \otimes v$

$$v^2 = 0 \qquad \underset{0}{\underbrace{(x+y)^2}} = x^2 + xy + yx + y^2$$
$$= xy + yx$$

So all $v^2 = 0 \Rightarrow xy + yx = 0$ all x, y.

if $xy + yx = 0$ all x, y, $\quad v = x = y \quad v^2 + v^2 = 0$
$$\Rightarrow 2v^2 = 0 \overset{?}{\Rightarrow} v^2 = 0$$

Ex: $V = Rx \oplus Ry$

$\Lambda(V)_0 = R \qquad \Lambda(V)_1 = V$

$\Lambda(V)_2 = \cancel{Rx^2} \oplus Rxy \oplus \cancel{Ry^2} = Rxy$

notation: write $x \wedge y$ for $x \otimes y$ instead of xy

$x \wedge y = -y \wedge x$

$\Lambda(V)_3 = 0 \qquad \Lambda(V)_i = 0 \quad i \geq 3.$

Notation: $\Lambda^i(V) \equiv \Lambda(V)_i \qquad S^i(V) \equiv S(V)_i$

$\otimes^i V = T(V)_i$

V = free w/ basis $x_1, \ldots, x_n$

$$\mathrm{rk}\left(\Lambda^i(V)\right) = \begin{cases} \binom{n}{i}, & i \in \{0, \ldots, n\} \\ 0 & \text{else} \end{cases}$$

All of these algebras $T(V)$, $S(V)$, $\Lambda(V)$ and modules $\otimes^i V$, $S^i(V)$, $\Lambda^i(V)$ are functorial in V.

i.e. given $V \longrightarrow W$ R, Mod map,

get induced maps $T(V) \to T(W)$

$T^i(V) \to T^i(W)$

$A(V) \to A(W)$

$C(V) \to C(W)$

$S(V) \to S(W)$ $\quad\quad$ $\Lambda(V) \to \Lambda(W)$

$S^i(V) \to S^i(W)$ $\quad\quad$ $\Lambda^i(V) \to \Lambda^i(W)$

i.e. $\varphi : V \longrightarrow W$

$\alpha = a_1 a_2 \cdots a_n \in S^n(V)$ $\quad$ $\varphi(\alpha) \equiv \underbrace{\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)}_{\in S^n(W)}$

If V is free w/ basis $e_1, \dots, e_n$

$\Lambda^n V$ is free of rank $1 = \binom{n}{n}$ w/ basis $e_1 \wedge \cdots \wedge e_n$

if $\varphi : V \longrightarrow V$ is an R-module map, get an induced map $\begin{array}{ccc}\Lambda^n V & \longrightarrow & \Lambda^n V \\ \psi \downarrow \cong & & \downarrow \cong \psi \\ R & & R\end{array}$

always has form mult. by an element of R.

"determinant"

$\det \varphi$ = induced map on $\Lambda^n V$ from φ.

$\varphi(e_i) = v_i$ $\qquad$ $\varphi(e_1 \wedge \dots \wedge e_n) = v_1 \wedge \dots \wedge v_n$

matrix for φ

$\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

by prop of Λ, alt. function of rows.
multilinear,
1 on id matrix.

Expand: Claim: induced map $\Lambda^n V \xrightarrow{\varphi} \Lambda^n V$ is mult. by some $\lambda \in R$.

Pf, consider $\varphi(e_1 \wedge \dots \wedge e_n) = \lambda e_1 \wedge \dots \wedge e_n$

$\Rightarrow \varphi(\mu e_1 \wedge \dots \wedge e_n) = \mu \varphi(e_1 \dots) = \mu \lambda \dots$

$= \lambda(\mu e_1 \wedge \dots \wedge e_n)$ $\square$

Modules over PIDs.

R a (commutative) principal ideal domain.

<u>Thm</u> If M is an R-module, free of finite rank ($M \cong R^n$ some n), $N < M$ submodule $\Rightarrow$ N is free of rank at most n.

<u>Pf</u>: choose $\{e_i\}$ basis for M, consider $K = \bigoplus_{i \geq 2} e_i R$

If $N' = N \cap K$, true for N' by induction.

Consider $\{r \in R \mid re_1 + k \in N, \text{ some } k \in K\} = I$

$$I \triangleleft R. \Rightarrow I = \partial R.$$

if $I = 0$ done by induction. else, choose $f_1 \in N$ s.t. $f_1 = \partial e_1 + k$, $k \in K$.

By induction, can choose $f_2, \ldots, f_l$, $l \leq n$ basis for N'. Claim: $f_1, f_2, \ldots, f_l$ basis for N.

Note if $\sum a_i f_i = 0 \Rightarrow a_1 = 0$ since

coeff of $e_1 = a_1 d$

and $\sum_{i \geq 2} a_i f_i = 0 \Rightarrow$ all rest of $a_i = 0$

since they are a basis

for N' by induction.

If $n \in N$, $n = a e_1 + n'$ $\quad a \in I$

$a = bd$

$$n - b f_1 = a e_1 + n' - (b d e_1 + \cancel{bk})$$

$= n' - bk \in N'$ in span $f_2, \ldots, f_e$

$n \in$ span f_i's.