Ch 12.1 (Modules over a PID)

Last time: If R a PID, $M < R^n \Rightarrow M$ is free of rank $\leq n$.

Main Thm: If R is a PID, M a f.g. R-module then M is a finite product (= direct sum) of cyclic R-modules.

Recall: if M an R-mod, M cyclic if $M = Ra$ some $a \in M$

In this case, get $R \twoheadrightarrow M$, $r \longmapsto ra$, $K = \ker$

$$K = Rd \trianglelefteq R \Rightarrow M \simeq R/\underset{\substack{\parallel \\ Rd}}{\langle d \rangle}$$

What one would like - canonical decomposition into cyclics.

e.g. $R = \mathbb{Z}$ $\quad M = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/40 \times \mathbb{Z}/8 \times \mathbb{Z}/2 \times \mathbb{Z}/9$

CRT: $\rightsquigarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/5 \times \mathbb{Z}/8 \times \mathbb{Z}/8 \times \mathbb{Z}/2 \times \mathbb{Z}/9$

unique expression: "elementary divisors"
(up to reordering)

2-primary
"2-power torsion"
3-p—

intermediate step

$$(\mathbb{Z}\times\mathbb{Z})\times\left(\mathbb{Z}/8\times\mathbb{Z}/8\times\mathbb{Z}/2\right)\times\left(\mathbb{Z}/9\right)\times\left(\mathbb{Z}/5\right)$$

primary components
"primary decomposition"

Invariant factor decomposition:

$$(\mathbb{Z}\times\mathbb{Z})\times\mathbb{Z}/a_1\times\mathbb{Z}/a_2\times\cdots\times\mathbb{Z}/a_n$$

$$a_1 \mid a_2 \mid \cdots \mid a_n$$

$8, 9, 5$

$a_{n-1} \leftarrow$ after elementary
8 highest left

$a_{n-2} \leftarrow 2$

$a_1 = 2 \quad a_2 = 8 \quad a_3 = 8\cdot 9\cdot 5$

$$M \cong (\mathbb{Z}\times\mathbb{Z})\times(\mathbb{Z}/2)\times(\mathbb{Z}/8)\times(\mathbb{Z}/8\cdot 9\cdot 5)$$

Proof of Main thm

M f.g. R-module

Define $\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ some } r \in R \setminus \{0\}\}$

Can choose generators $m_1, \ldots, m_\ell \in M$

$$R^\ell \twoheadrightarrow M$$

$$(r_1, \ldots, r_\ell) \longmapsto \sum r_i m_i$$

some kernel $K \subset R^\ell$, K free of rk $\leq \ell$.

Claim: Can find a basis $e_1, \ldots, e_\ell$ of R^ℓ such that $a_1 e_1, a_2 e_2, \ldots, a_s e_s$ basis for K $s \leq \ell$

some $a_i \in R$.

Note: if Claim true, done since $M \cong R^\ell / K$

$$\cong R/a_1 \times R/a_2 \times \cdots \times R/a_s \times \underbrace{R \times \cdots \times R}_{\substack{\ell - s \\ \text{copies of } R}}$$

(via $R^\ell \to R^\ell / K \cong M$)

Strategy: find elements of K which are "minimally divisible" $K < R^\ell$

Coords fcns $= \mathrm{Hom}_R(R^\ell, R)$

$\sum_r = \{ \varphi(k) \mid k \in K, \varphi \in \mathrm{Hom}_R(R^\ell, R) \} \triangleleft R$

let $a \in \sum_r$ be a generator, $y \in K$, $\pi \in \mathrm{Hom}_R(R^\ell, R)$

s.t. $\pi(y) = a$.

by construction $a \mid \varphi(z)$ all $\varphi \in \mathrm{Hom}_R(R^\ell, R)$
$z \in K$.

write $y = \sum a_i e_i$ e_i basis for R^ℓ standard

$a_i = \pi_i(y)$ π_i coord functions w/r/to e_i's.

$\Rightarrow a_i = d_i a$ some $d_i \in R$.

can divide and get $x = \sum d_i e_i$ $ax = y$.

<u>Claim</u>: x can be extended to a basis.

Consider $\varphi : R^\ell \longrightarrow R$

$$y \longmapsto a$$

$$x \longmapsto 1$$

Claim: $R^\ell = Rx \oplus (\ker \varphi)$

$$rx + k \longleftarrow (rx, k)$$

$$v \longmapsto (\varphi(v) \cdot x, (v - \varphi(v)x))$$

Claim': $K = Ry \oplus (\ker \varphi|_K)$

$$ry + k \longleftarrow (ry, k)$$

$$v \longmapsto \left(\underset{\overset{\shortparallel}{\varphi(v)x}}{\frac{\varphi(v)}{a} y}, \left(v - \frac{\varphi(v)}{a} y\right)\right)$$

+ induction

In summary:

$K < R^\ell$ found $y = ax \in K$ such that

$R^\ell \simeq Rx \oplus P$ $\quad (P = \ker \varphi)$

$K \simeq Ry \oplus (P \cap K)$ $\quad \mathrm{rk}\, P = \ell - 1$

continue w/ $P \hookrightarrow R^\ell$ (know P free)

$P \cap K \hookrightarrow K$

$$\Rightarrow M \cong R \times \dots \times R \times R/a_1^{n_1} \times R/a_2^{n_2} \times \dots \times R/a_m^{n_m}$$

a_i irred's in R.

(CRT)

rearranging factors, can get expression

$$M \cong R \times \dots \times R \times R/b_1 \times R/b_2 \times \dots \times R/b_n$$

$b_1 | b_2 | \dots | b_n$.

<u>More detail:</u>

$$R/aR = R/a_1^{n_1} \times \dots \times R/a_m^{n_m}$$

$a = a_1^{n_1} a_2^{n_2} \dots a_m^{n_m}$ a_i irred (UFD)

$$(a_i^{n_i}, a_j^{n_j}) = bR$$

$b | a_i^{n_i}$ & $b | a_j^{n_j}$

$\Rightarrow a_i = a_j$ or $b = 1$.
up to units

Uniqueness of presentations

Suppose $R \times \cdots \times R \times R/a_1 \times \cdots \times R/a_n = M$

$F = \text{frac}(R) \qquad M \otimes_R F$

$$M \otimes_R F \cong (R \otimes_R F) \oplus (R \otimes_R F) \cdots (\ ) \oplus (R/a_1 \otimes_R F) \cdots \oplus (R/a_n \otimes_R F)$$

$R \otimes_R F = F$

$$R/bR \otimes_R F = 0 \Longleftarrow \quad \bar{r} \otimes f = \bar{r} \otimes bb^{-1}f = \bar{r}b \otimes b^{-1}f = 0$$

$M \otimes_R F$ is an F-vectorspace

$\dim = \#$ of R's in decomposition.

"rank of M"

If $R/aR = M$ consider $M \otimes_R R/pR$

p irred.

$$R/aR \cong R/p_1^{n_1}R \times \cdots \times R/p_m^{n_m}R$$

$$R/p_i^{n_i} \otimes R/pR \simeq \begin{cases} 0 & \text{if } (p_i, p) = 1 \\ R/pR & \text{else} \end{cases}$$

$$(p_i, p) = 1 \Leftrightarrow (p_i^{n_i}, p) = 1$$

$$a p_i^{n_i} + bp = 1$$

$$R/p_i^{n_i} \otimes R/pR \ni \bar{r} \otimes \bar{s} = \bar{r} \otimes 1\bar{s} = \bar{r} \otimes (a p_i^{n_i} + bp)\bar{s}$$

$$= \bar{r} \otimes a p_i^{n_i} \bar{s} + \bar{r} \otimes bp\bar{s}$$

$$= \underbrace{\bar{r} p_i^{n_i}}_{0} \otimes a\bar{s} + \underbrace{\bar{r} \otimes bp\bar{s}}_{0}$$

$$R/p^n R \otimes_R R/pR = R/pR$$

$$(a \otimes b) \longmapsto \bar{a}b$$

$$1 \otimes a \longleftarrow a$$

$R/aR \otimes_R R/pR$ = unique v.s. over R/pR of dim = 1 if $p \mid a$

0 else.

consider pM