Examples / Applications of modules over a PID

$R = \mathbb{Z}$ $\quad M = \mathbb{Z}/20 \times \mathbb{Z}/25$

$$\mathbb{Z}/4 \times \mathbb{Z}/5 \times \mathbb{Z}/25$$

$$\mathbb{Z}/5 \times (\mathbb{Z}/100)$$

$R = \mathbb{C}[x]$ $\quad M = \mathbb{C}[x]/(x^6 - x^2)$

$$\mathbb{C}[x]/x^2(x^4-1)$$

$$\simeq \mathbb{C}[x]/x^2 \oplus \mathbb{C}[x]/x-1 \oplus \mathbb{C}[x]/x+1$$

$i \quad -i$

$R = \mathbb{R}[d]$ $\quad M = \{f \in \mathbb{R}[x] \mid \deg f \leq 5\}$

$$d \cdot f(x) \equiv \frac{d}{dx} f(x)$$

$$M \simeq \frac{\mathbb{R}[d]}{d^6} \qquad x^5 \text{ generator}$$

$$\begin{array}{ccc} \mathbb{R}[d] & \longrightarrow & M \\ 1 & \longmapsto & x^5 \end{array}$$

$$f(d) \longmapsto f(d)\cdot x^5$$

$\cdot 1$

$$a_0 + a_1 d + a_2 d^2 + \cdots + a_5 d^5 \longmapsto a_0 x^5 + a_1 5x^4 + a_2 20x^3 + \cdots$$

$R = \mathbb{R}[s]$ same M $\quad s \cdot f(x) = x f'(x)$

$$M \simeq \mathbb{R}[s]/s \times \mathbb{R}[s]/s-1 \times \cdots \times \mathbb{R}[s]/s-5$$

$R = \mathbb{R}[d]$ $\quad M = \{a \sin x + b \cos x \mid a, b \in \mathbb{R}\}$

$R = \mathbb{Q}[x]$ $\quad M = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$

as a $\mathbb{Q}$-v.space $M \simeq \mathbb{Q}^2$

where $x \cdot \lambda \equiv \sqrt{2}\lambda$

i.e. $x \cdot (a + b\sqrt{2}) = 2b + a\sqrt{2}$

ex: $M \simeq \mathbb{Q}[x]/x^2-2$

$R = \mathbb{Q}[x]$ $\quad M = \mathbb{Q}^3$

$$x \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left(f(x) \cdot g(x)\right) v = f(x)\left(g(x) v\right)$$

$$f(x)(v+w) = f(x)v + f(x)w$$

$$\left(f(x)+g(x)\right)(v) = f(x)v + f(x)w$$

$$1 \cdot v = v$$

$R \longrightarrow \text{End } M$ a ring hom.

From now on, consider V a vector space over a field F, $T : V \longrightarrow V$ is a linear transformation

In this case, V becomes an $F[x]$-module via $x \cdot v \equiv T(v)$

$f(x) = \sum a_i x^i$, then $f(x) \cdot v \equiv \sum a_i T^i(v)$

Consequently, V a sum of cyclic modules (if f.gen!) assume $\dim_F V < \infty$

$$V \cong \cancel{F[x]^n} \times \frac{F[x]}{f_1} \times \cdots \times \frac{F[x]}{f_m}$$

∞-dim'l

<u>Goal</u>: describe the action of T on V via f_i's.

<u>Warmup</u>: suppose $V \cong \frac{F[x]}{f}$ $\quad f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$

F-basis: $1, x, x^2, \ldots, x^{n-1}$

how does T act w/rt to this basis?

mult. by x:
$$1 \longrightarrow x$$
$$x \longmapsto x^2$$
$$\vdots$$
$$x^{n-2} \longrightarrow x^{n-1}$$
$$x^{n-1} \longmapsto x^n = -a_{n-1}x^{n-1} - a_{n-2}x^{n-2} \cdots - a_0$$

this shows: can choose a basis to V
(corresp to $1, x, \dots, x^{n-1}$)
such that matrix for T looks like

$$T \longleftrightarrow \begin{bmatrix} 0 & 0 & & -a_0 \\ 1 & 0 & & \vdots \\ 0 & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & -a_{n-2} \\ 0 & 0 & 1 & -a_{n-1} \end{bmatrix} = C_f \quad \text{"companion matrix" for } f$$

in general: as a $F[x]$-module $V \cong \frac{F[x]}{f_1} \times \dots \times \frac{F[x]}{f_m}$

$\Rightarrow$ can find a decomp of $V \cong V_1 \times \dots \times V_m$
where $V_i \cong \frac{F[x]}{f_i}$ as $F[x]$-mods (via action of T)
choosing bases of V_i's as above, get

$$T \longleftrightarrow \begin{bmatrix} C_{f_1} & & 0 \\ & C_{f_2} \ddots & \\ 0 & & C_{f_m} \end{bmatrix} \quad \text{block-diagonal.}$$

if we use invariant factor decomp. f_i monic
$V \cong F[x]/f_1 \times \dots \times F[x]/f_m \quad f_1 | f_2 | \dots f_m$

gives a canonical presentation for matrix for T

called "rational canonical form"

These describe in a complete way, the set of matrices up to similarity (= change of basis)

Shortcoming – lack of transparency

e.g. $T \longleftrightarrow F[x] / (x-a)^3 (x-b)$

$$\simeq F[x]/(x-a)^3 \times F[x]/x-b$$

$V \simeq W \times U$ $\quad$ U 1-dim'l eigspace w/ val b.

generalized e-space w/ value a

$(T-a)^3$ acts as 0

basis $e_1 \to e_2 \to e_3 \to 0$
by $(T-a)$

Reminder

Def $v \in V$ is a generalized eigenvector for T w/ eigenvalue λ means $(T-\lambda)^m v = 0$ some $m > 0$

[note if $(T-\lambda)^{m-1} v \neq 0$ then $(T-\lambda)^{m-1} v$ an evector w/ val λ.

Note: Companion matrix for $x-\lambda$ is $[\lambda]$

So, if $V = F[x]/_{x-\lambda_1} \times F[x]/_{x-\lambda_2} \cdots$

companion block diagonal form = diagonal matrix

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

to find evects:

Solve $Tv = \lambda v$ $\quad (T - \lambda \mathrm{Id}) v = 0$

Solve $\underbrace{\det(\lambda I - T)}_{\text{monic poly in } \lambda} = 0$

Def $\chi_T(x) = \mathrm{char}_T(x) = \det(xI - T)$

deg n poly $n = \dim V$

roots $\longleftrightarrow$ e. values

$V \cong \frac{F[x]}{f_1} \times \cdots \times \frac{F[x]}{f_m}$ w/r/t to action by $x \leftrightarrow T$

$$\Rightarrow T \longleftrightarrow \begin{bmatrix} C_{f_1} & & 0 \\ & \ddots & \\ 0 & & C_{f_m} \end{bmatrix}$$

$$\det(xI - T) = \det \begin{bmatrix} x - C_{f_1} & & & \\ & x - C_{f_2} & & 0 \\ & 0 & \ddots & \\ & & & x - C_{f_m} \end{bmatrix}$$

$$= \prod \det(xI - C_{f_i}) = \prod f_i$$

$$\det(x - C_f) = \det \begin{bmatrix} x & & 0 & a_0 \\ -1 & \ddots & & \vdots \\ & \ddots & \ddots & a_{n-2} \\ 0 & & -1 & x + a_{n-1} \end{bmatrix}$$

$= f(x)$

Prop $\chi_T(x) = \prod f_i$ w/r/to any cyclic decomp of $V \cong \times \frac{F[x]}{f_i}$

ex: $\chi_T(x) = \prod (x-\lambda_i)$ λ_i distinct

$\Rightarrow$ elm. divisr decom. must be

$V \cong \times F[x]/_{x-\lambda_i}$ $\Rightarrow \exists$ basis where

$$T \leftrightarrow \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Def $m_T(x) = \min_T(x)$

If we set $F[x] \longrightarrow \operatorname{End} V$
$x \longmapsto T$

ring hom, kernel is principal w/ some generatr.

Def: $m_T(x) =$ the monic generatr of this kernel.

$\Longrightarrow$ $m_T(T)$ acts as 0 on V
& if $g(T) = 0$ on $V \Rightarrow m_T \mid g$

Thm $m_T(x) = \mathrm{lcm}\{f_i\}$ w/r/to any cyclic decomp

Pf: want to show $g(T) = 0 \overset{!}{\Leftrightarrow} f_i \mid g$ all i

$$g(T) = 0 \Leftrightarrow m_T \mid g$$

$$V \cong \bigtimes_i F[x]/f_i \qquad W_i \leftrightarrow F[x]/f_i, \quad W_i \subseteq V$$

T preserves W_i's $\quad g(T)$ preserves W_i

$g(T) = 0$ on $V \Leftrightarrow g(T) = 0$ on each $W_i \cong F[x]/f_i$

$$\Leftrightarrow f_i \mid g \qquad \square.$$

$$\chi_T(x) = \prod f_i \qquad m_T(x) = \mathrm{lcm}\{f_i\}$$

$$\Rightarrow m_T \mid \chi_T$$

Cor m_T & χ_T have the same irreducible factors

π irred in $F[x]$ then $\pi \mid m_T \Leftrightarrow \pi \mid \chi_T$

Pf: $\pi | m_T | \chi_T \Rightarrow \pi | \chi_T$

if $\pi | \chi_T \Rightarrow \pi | f_i$ some $i \Rightarrow \pi | \operatorname{lcm}\{f_i\} = m_T$ $\square$.

Jordan form

Suppose T as above, $\chi_T(x)$, can factor into linear factors in $\mathbb{F}[x]$

$$\chi_T(x) = \prod (x-\lambda_i)^{n_i} \quad \lambda_i \in F.$$

If $\lambda_i \in F \Rightarrow$ this factorization occurs in $F[x]$

in this case,

$$V \simeq F[x]/(x-\lambda_1)^{n_1} \times \cdots \times F[x]/(x-\lambda_m)^{n_m}$$

(= elem divisor decomp)

if $T \longleftrightarrow F[x]/(x-\lambda)^n$ $\quad \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix} = S$

claim: this matrix gives same cyclic form

$F[x]/(x-\lambda)^n$

why does this work?

$(S-\lambda)^n = 0$ on V

$(S-\lambda)^{n-1} \neq 0$ on V

" $[0']$ $\Rightarrow m_S \mid (S-\lambda)^n$

$m_S \nmid (S-\lambda)^{n-1}$

$\Rightarrow m_S = (S-\lambda)^n$

$m_S \mid \chi_S = \deg n \Rightarrow \chi_S = (S-\lambda)^n$

cyclic decomp. for S $\quad \times \dfrac{F[x]}{f_i}$

$\prod f_i = (S-\lambda)^n$

$\operatorname{lcm} f_i = (S-\lambda)^n$

$\Rightarrow$ one f_i, $= (S-\lambda)^n$

cyclic expression $= \dfrac{F[x]}{(S-\lambda)^n}$

$$V \cong X \frac{F(x)}{(x-\lambda_i)^{n_i}}$$

$$T \longleftrightarrow \begin{bmatrix} \lambda_1 & 1 & & & & & & & \\ & \ddots & 1 & & & & & 0 & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & 1 & & & & \\ & & & & \ddots & 1 & & & \\ & 0 & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_m & 1 \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_m \end{bmatrix}$$

$n_2 \times n_2$ block

$n_m \times n_m$ block

$$\chi_T(x) = (x-2)^3$$

$F[x]/(x-2)^2$ $\qquad$ $F[x]/x-2$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 2 \end{bmatrix}$$

$m_T = (x-2)^3$ $\qquad$ $\frac{F[x]}{x-2}$ $\qquad$ $m_T = x-2$

$m_T = (x-2)^2$