Jordan–Hölder

X-Groups

Let X be a set.

Def an X-group is a group G, together with a map of sets $X \xrightarrow{\varphi} \operatorname{End} G = \operatorname{Hom}_{gp}(G, G)$

notation $x \cdot g \equiv \varphi(x)(g)$

Axiomatically:

$$x \cdot (gh) = (x \cdot g)(x \cdot h)$$

G abelian $\quad x \cdot (g+h) = x \cdot g + x \cdot h$

Def G, H are X-gps, a X-gp hom $f : G \to H$ is a group hom s.t. $f(x \cdot g) = x \cdot f(g)$ all $x \in X$, $g \in G$.

Def sub X-gp, normal sub X-gps

X-closed subgp.

" ker of X-gp hom

Standard gp isom thms hold

Ex

- R ring, M an R-mod then M is a R-gp
- If G is a group, it is an $\emptyset$-gp

M, N R-mods, then R-mod Homs from M to N
= R-gp " —— "

$f: M \to N$
$f(rm) = r f(m)$
$f(m + m') = f(m) + f(m')$

Def a nontrivial X-gp G is <u>simple</u> if no nontrivial normal sub X-gps.

Def An X-gp G has <u>finite length</u> if $\exists$ a sequence of sub. X-gps "composition sequence"

$(e) = G_0 \subset G_1 \subset \dots \subset G_n = G$

with $G_{i-1} \triangleleft G_i$ and G_i/G_{i-1} simple.

given sequence as above, the gps G_i/G_{i-1} are the "composition factors", n = length

Thm (Jordan-Hölder Theorem) G an X-group

If G has finite length, then any two composition sequences have same length, and composition factors are same up to reordering.

Pf:

Pf by induction on min'l length of a comp. series for G

if length $= 1$ done ✓

consider two comp. series, with first minimal.

$(e) \subset G_0 \subset \cdots \subset G_n = G$

$(e) \subset H_0 \subset \cdots \subset H_m = G$

consider $K = G_{n-1} \cap H_{m-1} \triangleleft G$

case 1: $G_{n-1} = H_{m-1}$ done by induction via G_{n-1}
($\Rightarrow n-1 = m-1$, same facts up to $G_{n-1} = H_{m-1}$ ...)

Since $G_{n-1} \triangleleft G$ $H_{m-1} \triangleleft G$

consider images of G_{n-1} & H_{m-1} in G/K

images both nontrivial

$$G_{n-1}/K = G_{n-1}/H_{m-1} \cap G_{n-1} \simeq G_{n-1}H_{m-1}/H_{m-1}$$

$$\simeq G/H_{m-1}$$

simple

since
$G_{n-1} \neq H_{m-1}$
both maximal normal

H_{m-1}/K simple

Choose a comp series for K

$(e) = K_0 \subset K_1 \subset \dots \subset K_s = K$

$$K_0 \subset K_1 \subset \dots \subset \overset{K}{\overset{\shortparallel}{K_s}} \subset G_{n-1} \subset G$$

also a comp series.

$\Rightarrow K_0 \subset \dots \subset K_s \subset G_{n-1}$ comp series

by induction hyp since G_{n-1} has a shorter series than G

this sequence has length $n-1$

[☆ need to justify : K has finite length]

and comp factors of $K_0 \subset \dots \subset K_s \subset G_{n-1}$

same as in $G_0 \subset \dots \subset G_{n-2} \subset G_{n-1}$

consider $K_0 \subset \dots \subset K_s \subset H_{m-1} \subset H_m$

$s = n-2$

$\Rightarrow K_0 \subset \dots \subset K_s \subset H_{m-1}$ length $n-1$

$\Rightarrow H_{m-1}$ has length $< n$

$\Rightarrow H_{m-1}$ has well defined length & unique factors

$\Rightarrow m-1 = n-1 \Rightarrow n = m$

comp factors of $H_0 \subset \cdots \subset H_{n-1} \subset H_n = G$

same factors $K_0 \subset \cdots \subset K_s \subset H_{n-1} \subset H_n = G$

same factors $K_0 \subset \cdots \subset K_s \subset G_{n-1} \subset G = G$

$G_0 \subset \cdots \subset G_{n-1} \subset G = G$

and done since $H_n/H_{n-1} \cong G_{n-1}/K$

$G_n/G_{n-1} \cong H_{n-1}/K$ $\square$

For unjustified Claim

if G has finite length, $K \triangleleft G$

then K has finite length ime

Start w/ $G_0 \triangleleft \cdots \triangleleft G_n$ ⤳ intersect w/ K

$$(G_0 \cap K) \triangleleft (G_1 \cap K) \triangleleft \cdots \triangleleft (G_n \cap K)$$

$$G_i/G_{i-1} \triangleright (G_i \cap K)G_{i-1}/G_{i-1} \cong G_i \cap K / G_{i-1} \cap K$$

simple

$\Rightarrow$ $G_{i}\cap K / G_{i-1}\cap K$ is simple or trivial.

so get a comp series by erasing extra terms in sequence

Cor J-H for finite gps

Cor J-H for modules of finite length over a ring
"
Art + Noeth.

And now for something completely different

Def A category $\mathcal{C}$ consists of a collection of objects $ob(\mathcal{C})$, and for any pair of objects $X, Y \in ob(\mathcal{C})$ a set of morphisms $Hom_{\mathcal{C}}(X,Y)$

together with:

- for each $X \in ob(\mathcal{C})$ a distinguished morphism $1_X \in Hom_{\mathcal{C}}(X,X)$
- a comp rule $Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z)$

$\mathrm{Hom}_{\mathcal{C}}(X,Z)$

s.t. Axioms:

(associativity, identities)

Def Opposite Category

Def A functor $F: \mathcal{C} \to \mathcal{D}$ is a rule which associates to each object $X \in \mathcal{C}$ an object $FX \in \mathcal{D}$ & a collection of maps $\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow[F]{} \mathrm{Hom}_{\mathcal{D}}(FX,FY)$

$f: X \to Y \qquad Ff: FX \to FY$

s.t. $F(fg) = F(f)F(g)$

$F(1_X) = 1_{FX}$

Def: $F, G: \mathcal{C} \to \mathcal{D}$ functors, a nat trans $\alpha: F \Rightarrow G$ is a rule which associates to each $X \in \mathcal{C}$ a morphism

$\alpha_X : F(X) \to G(X)$

i.e. $\alpha_X \in \mathrm{Hom}_{\mathcal{D}}(FX, GX)$

s.t. if $X \xrightarrow{f} Y$, then the digram

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\ GX \arrow[r, "Gf"] & GY \end{tikzcd}$$ commutes.

<u>Ex:</u> Sets, Gps, $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ are cats

objects = functors

morphisms

$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F,G)$

$=$

$\{\text{nat trans } \alpha : F \Rightarrow G\}$

<u>Ex</u> $\mathcal{C}$ a cat, $X \in \mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}}(-,X)$ functor $\mathcal{C}^{op} \to \text{Sets}$

$\mathrm{Hom}_{\mathcal{C}}(X,-)$ functor $\mathcal{C} \longrightarrow \text{Sets}$

<u>Ex:</u> $\mathcal{C}$ cat

$$\mathcal{C}^{op} \longrightarrow \mathrm{Fun}(\mathcal{C}, \text{Sets})$$
$$X \longmapsto \mathrm{Hom}_{\mathcal{C}}(X,-)$$

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets}) \\ X &\longmapsto \mathrm{Hom}_{\mathcal{C}}(-, X) = h_X \end{aligned}$$

<u>Thm</u> (Yoneda lemma)

for any cat $\mathcal{C}$, the functor $h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$
$X \mapsto h_X$

is fully faithful

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow[h]{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})}(h_X, h_Y)$$