Functors equivalent to the identity

if $F: \mathcal{C} \to \mathcal{C}$ is a functor, $\alpha : F \overset{\sim}{\Rightarrow} 1_{\mathcal{C}}$ = identity

means, for each object $X \in \mathcal{C}$, choose $\alpha_X : FX \xrightarrow{\sim} X$

such that $\forall X \xrightarrow{f} Y$ in $\mathcal{C}$, have a commutative diagram

$$\begin{array}{ccc} FX & \xrightarrow{\alpha_X} & X \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle f} \\ FY & \xrightarrow[\alpha_Y]{} & Y \end{array}$$

"relabeling" or "alternate choice" of objects & morphisms.

ex: $\mathcal{C} = \underline{\text{Groups}}$

FG = new gp w/ underlying set

$\{(1,g) \mid g \in G\}$

$(1,g)(1,h) = (1,gh)$

$FG \xrightarrow{\sim} G$

Definitions

If $F: \mathcal{C} \to \mathcal{D}$ is a functor, get, for each

$X, Y \in \mathcal{C}$, $\quad F: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$

we say F is <u>faithful</u> if ↑ is injective, all X, Y, we say F is **full** if $\diagup$ is surjective, all X, Y

F is **essentially surjective** if $\forall\, y \in \mathcal{D}\ \exists\, X \in \mathcal{C}$ s.t.

$FX \simeq y$ in $\mathcal{D}$.

Def F is an **equiv. of categories** if

FG & GF are isom. to the identity (of $\mathcal{D}$ & $\mathcal{C}$)

Lem F is an equiv. of cats $\Longleftrightarrow$ F is fully faithful & essentially surjective.

Pf: For $d \in \mathcal{D}$, choose $c(d) \in \mathcal{C}$ & an isom $\quad G = c : ob(\mathcal{D}) \to ob(\mathcal{C})$

$\varphi_d : d \xrightarrow{\sim} Fc(d)$, define $Gd = c(d)$

and for $d \xrightarrow{g} d'$

$$\begin{tikzcd} d \arrow[r, "\sim", "\varphi_d"'] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowright"] & Fc(d) \arrow[d, "\exists!"] & & c(d) \arrow[d, "Gg \leftarrow \mathrm{def}"] \\ d' \arrow[r, "\sim", "\varphi_{d'}"'] & Fc(d') & & c(d') \end{tikzcd} \qquad \xleftarrow[\sim]{F}$$

now check $FG, GF \simeq 1$. $\square$

Yoneda

Set $\widehat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$

$$h : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$$
$$y \longmapsto h_y = \mathrm{Hom}_{\mathcal{C}}(-, y)$$

Claim: h is fully faithful.

Will show:

for $F \in \widehat{\mathcal{C}}$, define $\widehat{F} \in \widehat{\mathcal{C}}$ via

$$\widehat{F}(X) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, F)$$

$$\widehat{F}(-) = \mathrm{Hom}_{\widehat{\mathcal{C}}}(\mathrm{Hom}_{\mathcal{C}}(-, X), F)$$

$$\widehat{F} = \mathrm{Hom}_{\widehat{\mathcal{C}}}(-, F) \circ h$$

there is a nat. trans $\alpha : F \to \widehat{F}$

for $X \in \mathcal{C}$, $\alpha_X : FX \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_X, F)$

given $a \in FX$, want $h_X \Rightarrow F$

i.e. for $Y \in \mathcal{C}$ $\quad h_X(Y) \to F(Y)$

i.e. $f: Y \to X$, want an elmt of $F(Y)$

have $a \in FX$, apply F to f get

$$\underset{\substack{\Psi \\ a}}{Ff: FX} \to FY \quad \text{get } Ff(a) \in F(Y)$$

<u>Conclusion</u> $\forall F \in \hat{\mathcal{C}}$, $\mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F) \simeq F(X)$

in particular, if $F = h_Y$ we get

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, h_Y) = h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

$\square$.

Representations of finite groups

Given a finite group G and a vector space V over a field F, we define a representation of G on V to be an (linear) action of G on V such that

- $g(v+w) = gv + gw$
- $g(\lambda v) = \lambda g(v)$

$\Big\}$ "linearity"

<u>Alternately</u> it is a group hom $G \to \mathrm{Aut}_F(V) = GL(V)$

<u>Notation</u>: (V, ρ) $\rho : G \to GL(V)$
or just V (ρ implied)

$$\rho(g)(v) = gv$$

Given reps V, W of G, a hom. of reps is a lin map $\varphi : V \to W$ s.t. $\varphi(g \cdot v) = g \cdot \varphi(v)$
"G-map"

If V is a rep of G, $W < V$ a subspace is called a sub-representation if $GW < W$

(also called "invariant subspace")

if $W < V$ invariant, can define a G-action on quotient V/W by acting on reps

Can define $V \oplus W$ for reps V, W via

$$g(v,w) = (gv, gw)$$

We say that V is <u>indecomposable</u> if $V \simeq W_1 \oplus W_2$ $\Rightarrow W_1$ or $W_2 = 0$.

We say that V is <u>irreducible</u> if V has no nontrivial invariant subspaces (simple)

We say that $V \simeq W$ if $\exists$ bijective hom.

Recall group ring $FG = \{ \sum a_g g \mid a_g \in F,\ g \in G \}$

$$(ag)(bh) = ab(gh)$$

if V is a G-representation, V has a FG-mod structure

via $\left(\sum a_g g\right) v = \sum a_g\, g \cdot v$

moreover, if W is an FG-mod, W affords

a representation of G via $g \cdot w = g \cdot w$
FG struct.

$$G \hookrightarrow (FG)^*$$

and get an equiv. of cats $FG\text{-mod} \cong G\text{-reps}$.

<u>Def</u> a rep V of G is decomposable if it is not indecomp. It is <u>completely decomposable</u> if $V \cong \bigoplus W_i$, W_i irreducible.

(modules: semisimple)

(irrep = irreducible representation)

Maschke's Theorem

Suppose char $F \nmid |G|$, an V a f.d.im'l rep. of G, and $W < V$ a proper subrep. Then $\exists P < V$ inv. subspace s.t. $V \simeq W \oplus P$.

Cor: V completely decomposable.

Pf: induct on dim V. dim 1 ✓

if $W < V$ proper, minimal $\Rightarrow W$ irred.

$V \simeq W \oplus P$, P by induction is a sum of irreps

$\Rightarrow V$ is also $\square$

Pf: (of Maschke)

Choose any v.space decomp $V \simeq W \oplus \tilde{P}$

$\tilde{\pi} : V \longrightarrow W$ projection.

set $\pi(v) = \frac{1}{|G|} \sum_{g \in G} g(\tilde{\pi}(g^{-1}v))$

now, for $w \in W$, $\tilde{\pi}(w) = w$ and $g^{-1}w \in W$

$$\pi(w) = \frac{1}{|G|} \sum_{g \in G} g(\tilde{\pi}(\underbrace{g^{-1}w}_{W}))$$

$$= \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w$$

$$= \frac{|G|}{|G|} w = w.$$

$$\pi(hv) = \frac{1}{|G|} \sum_{g \in G} g(\tilde{\pi}(g^{-1}hv))$$

$$= \frac{1}{|G|} \sum_{\substack{hg \\ g \in G}} (hg)(\tilde{\pi}((hg)^{-1}hv))$$

$$= \frac{1}{|G|} \sum_{g \in G} hg(\tilde{\pi}(g^{-1}h^{-1}hv))$$

$$h\left(\frac{1}{|G|} \sum_{g \in G} g(\tilde{\pi}(g^{-1}v))\right) = h\pi(v)$$

now $P = \ker \pi$ is a sub representation.

$$V = P \oplus W \quad \text{via} \quad v \longmapsto \big(\underset{\substack{\in \\ P}}{v - \pi(v)}, \underset{\substack{\in \\ W}}{\pi(v)}\big)$$

$\square$