Last time:

If $\operatorname{char} F \nmid |G|$ then any rep. of G is completely reducible. (Maschke's thm)

R any ring (FG), M is simple (irred)

M is cyclic, since if $m \in M$, $m \neq 0$ $Rm < M$

$\Rightarrow Rm = M$ $\qquad R \twoheadrightarrow M$, $r \longmapsto rm$

$\Rightarrow M \simeq R/I$ some (maximal) left ideal $I \trianglelefteq_{\ell} R$.

if R mods have complements ($=$ semisimple (!))

$\Rightarrow I$ has a complement in R

$$R \simeq I \oplus M'$$

but $\underset{\overset{\parallel}{M}}{R/I} \cong M'$

$\Rightarrow$ any simple mod is a summand of R!

in our case, FG f.dim'l $\Rightarrow$

FG $\simeq \bigoplus$ finite # of simples,

and these must include all

possible simple FG-mods.

Structure of FG

as an FG-module, can write

$$FG \simeq \left(\bigoplus^{n_1} M_1\right) \oplus \left(\bigoplus^{n_2} M_2\right) \cdots \oplus \left(\bigoplus^{n_\ell} M_\ell\right)$$

M_i's distinct (non isom) simple FG-modules.

$$FG = \mathrm{Hom}_{FG\text{-mod}}(FG, FG)$$

$$= \mathrm{Hom}_{FG}\left(\left(\bigoplus M_1\right) \oplus \left(\bigoplus M_2\right) \cdots \oplus \left(\bigoplus M_\ell\right), \left(\bigoplus M_1\right) \cdots \oplus \left(\bigoplus M_\ell\right)\right)$$

$$R = \mathrm{Hom}_{R\text{-}R}(R, R)$$

$$\mathrm{Hom}_R(A \oplus B, A \oplus B) = \begin{bmatrix} \mathrm{Hom}(A,A) & \mathrm{Hom}(B,A) \\ \mathrm{Hom}(A,B) & \mathrm{Hom}(B,B) \end{bmatrix}$$

$$= \left\{ \begin{matrix} A \to A & B \to A \\ A \to B & B \to B \end{matrix} \right\}$$

$$FG \cong \left\{ \begin{bmatrix} \mathrm{Hom}(\oplus M_1, \oplus M_1) & & \\ \mathrm{Hom}(\oplus M_1, \oplus M_2) & & \\ \vdots & & \\ \mathrm{Hom}(\oplus M_1, M_l) & - & - \end{bmatrix} \right\}$$

entries: $\mathrm{Hom}(\oplus M_i, \oplus M_j)$

$$\begin{bmatrix} \mathrm{Hom}(M_i, M_j) & - & - & \mathrm{Hom}(M_i, M_j) \\ \vdots & & & \\ & & & \end{bmatrix}$$

if M_i, M_j simple

$\varphi : M_i \to M_j$ $\quad$ φ either 0 $\quad$ $\ker\varphi = M_i$

or φ is injective

$\ker\varphi = 0$

φ surj.

$\Rightarrow \varphi$ is an iso.

<u>Lem</u> (Schur's lemma) $\mathrm{Hom}(M_i, M_j) = \begin{cases} 0 & \text{if } M_i \neq M_j \\ \text{division ring} & \text{if } M_i = M_j \end{cases}$

(algebra)

$$\Rightarrow FG \cong \left\{ \begin{bmatrix} M_{n_1}(D_1) & & & 0 \\ & M_{n_2}(D_2) & & \\ & & \ddots & \\ 0 & & & M_{n_\ell}(D_\ell) \end{bmatrix} \right\}$$

$$\boxed{\begin{array}{l} \Rightarrow FG \cong M_{n_1}(D_1) \times \cdots \times M_{n_\ell}(D_\ell) \\ \qquad D_i = \mathrm{End}_{FG}(M_i) \text{ is a division algebra.} \end{array}}$$

Factors corresp. to irreps of G.

$$z_i = (0, \ldots, 0, 1, 0, \ldots 0)$$

1 in ith Matrix alg $M_{n_i}(D_i)$

$\sum z_i = 1$ $\qquad z_i^2 = z_i$ $\qquad z_i z_j = 0$ if $i \neq j$

$z_i \in Z(FG)$

z_i's are actually an F-basis for $Z(FG)$ ... later.

what are these D's? D_i's are f.dim'l division alg /F

if $F = \mathbb{R}$ $\rightsquigarrow$ $D = \mathbb{R}$

$\rightsquigarrow$ $D = \mathbb{C}$

$\rightsquigarrow$ $D = \mathbb{H}$

if $F = \mathbb{Q}$ (# field) $\rightsquigarrow$ D can be lots of diff't things.

if $F = \mathbb{C}$ $\rightsquigarrow$ $D = \mathbb{C}$

$D/\mathbb{C}$ f.dim'l $\quad \partial \in D \quad \mathbb{C}[\partial] \subset D$

$\uparrow$ domain

comm.

$\Rightarrow$ field ext. of $\mathbb{C} = \mathbb{C}$

$\Rightarrow D = \mathbb{C}.$

So $\mathbb{C}G \simeq \bigtimes_{i=1}^{l} M_{n_i}(\mathbb{C})$

each factor corresp. to an iso class of irrep.

$\Updownarrow$

simple $\mathbb{C}G$-mod.

what is it?

it is an $M_{n_i}(\mathbb{C})$-module.

$\boxed{\mathbb{C}^{n_i}}$ ← unique simple mod.

Cor for a finite gp G, have a finite list of irreps

$M_1, \dots, M_l$ s.t. $\dim_{\mathbb{C}} M_i = n_i$

$$\sum_{i=1}^{l} n_i^2 = \dim \mathbb{C}G = |G|$$

Prop: $\dim Z(\mathbb{C}G) = \#$ irreps

$= \#$ conj classes of elmts in G

Pf: Step 1: $\dim Z(\mathbb{C}G) = \#$ irreps

$$Z(\mathbb{C}G) = Z\left(\times M_{n_i}(\mathbb{C})\right)$$

$$= \times Z(M_{n_i}(\mathbb{C})) = \overset{\ell}{\times}\, \mathbb{C}$$

$\left(Z(M_{n_i}(\mathbb{C})) = \mathbb{C}\,\mathrm{Id} \right)$

if $K_1, \dots, K_r$ = conj. classes in G

set $X_i = \sum_{g \in K_i} g$. Claim: X_i's are a basis for $Z(\mathbb{C}G)$.

indep ✓ span?

if $X \in Z(\mathbb{C}G)$, $X = \sum c_g g$

then if $h \in G \subset \mathbb{C}G$ $\quad hXh^{-1} = X$

$$X = hXh^{-1} = h\left(\sum c_g g\right)h^{-1} = \sum_g c_g hgh^{-1}$$

$$= \sum_{\substack{h^{-1}gh \\ g \in G}} c_{h^{-1}gh}\, h(h^{-1}gh)h^{-1}$$

$$= \sum_{g \in G} c_{h^{-1}gh}\, g$$

$$= \sum_{g \in G} c_g g$$

$c_g = c_{h^{-1}gh}$ $\quad c_{\square} : G \to \mathbb{C}$ is constant on conj classes.

$\Rightarrow \chi$ is in span χ_i's. $\square$.

Part 2: Characters & Class Functions

Given a rep (V, ρ) of G / $\mathbb{C}$

Def χ_ρ "Character of ρ"

$\chi_\rho : G \to \mathbb{C}$ $\quad \chi_\rho(g) = tr(\rho(g))$

doesn't depend on a choice of basis.

Def A class function $f : G \to \mathbb{C}$ is a function s.t. $f(g) = f(g')$ whenever g & g' are conjugate.

Characters are class functions

$$\chi_\rho(hgh^{-1}) = tr(\rho(hgh^{-1}))$$
$$= tr(\rho(h)\rho(g)\rho(h)^{-1})$$
$$= tr(\rho(g)) = \chi_\rho(g)$$

In fact

Thm The characters of irreps of G form an (orthonormal) basis for all class functions on G.

Inner product (Hermitian)

$f_1, f_2 : G \longrightarrow \mathbb{C}$

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

Observations

(ρ, V) rep of G / $\mathbb{C}$

$$\chi_\rho(1) = tr(\rho(1)) = tr(1) = \dim V$$

useful rep: regular representation

V has basis $\longleftrightarrow$ elmts of G

$g \in G$ permutes basis by left mult.

$$\dim V = |G| \quad (V = \mathbb{C}G)$$

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases} \quad \text{(no diag entries)}$$

z_i — orthogonal idempotents in $\mathbb{C}G$ from before.

$z_i z_j = 0$ if $i \neq j$ $\qquad z_i^2 = z_i$

$$z_i \longleftrightarrow M_{n_i}(\mathbb{C}) \text{ factor} \longleftrightarrow \text{irrep } \rho_i \rightsquigarrow \chi_i$$

Prop: $\boxed{13}$ ch18

$$z_i = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g^{-1})\, g$$

Strategy:

$$z_i = \sum_{g \in G} a_g\, g$$

to find a_g, mult by $g^{-1} \rightsquigarrow z_i g^{-1}$

coeff of 1 in this $= a_g$

now apply reg rep to this $z_i g^{-1}$

$$\chi_{rg}(z_i g^{-1}) = |G| a_g$$

$$\rho_{rg} = \underbrace{\rho_1 \oplus \cdots \oplus \rho_\ell}_{\text{irreps}} \qquad \chi_{rg} = \sum \chi_i$$

$$\chi_i(z_j) = \begin{cases} 0 & \text{if } i \neq j \\ \dim V_i & \text{if } i = j \end{cases}$$