Burnside's Thm

If G is finite group of order $p^a q^b$ for primes p, q, then G is solvable.

Pf:

if $a=0$ or $b=0$ $\Rightarrow$ G is Nilpotent $\Rightarrow$ G is solvable.

If false, choose G s.t. $|G| = p^a q^b$ minimal

i.e. G is a counterexample.

If $N \triangleleft G$ nontrivial $\Rightarrow$ hyp. apply to $N, G/N$

$\Rightarrow N, G/N$ solvable $\Rightarrow$ G solvable.

WLOG, can assume G is simple, nonab.

Let $P \in Syl_p(G)$, P a p-gp, $Z(P)$ nontrivial

$\Rightarrow \exists g \in Z(P)$ $g \neq e$. Let K: conj class of g

$$|K| = \frac{|G|}{|C_G(g)|} \qquad g \in Z(P) \Rightarrow P \subset C_G(g)$$

$$\Rightarrow |K| = q^{b'}$$

this will contradict the follwg lemma:

Lemma if G is finite, K a conj class $K \neq \{e\}$ and $|K| =$ prime power p^c, $c > 0$. $\Rightarrow$ G cannot be simple, nonabelian.

We'll need the fact: characters of irreps are orthonormal:

χ_i chr φ_i irrep $\quad i = 1, \ldots, r$

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}$$

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}$$

Let $\varphi = \varphi_i$ an irrep $K = K_j$ conj. classes.

$\varphi : G \to GL(V)$

define $\Omega_{i,j} = \Omega_{\varphi, K} = \sum_{g \in K} \varphi(g)$

notice: if $h \in G$, $\varphi(h)$ commutes w/ $\Omega_{\varphi, K}$

$$\varphi(h)\,\Omega_{\varphi,K}\,\varphi(h)^{-1} = \sum_{g\in K} \varphi(h)\varphi(g)\varphi(h^{-1})$$

$$= \sum_{g\in K} \varphi(hgh^{-1}) = \sum_{g\in K} \varphi(g) = \Omega_{\varphi,K}$$

$$\begin{tikzcd} \mathbb{C}G \simeq \times M_{n_i}(\mathbb{C}) \arrow[r, "\pi_i"] & M_{n_i}(\mathbb{C}) \\ G \arrow[u, hook] \arrow[r, "\varphi"] & GL_{n_i}(\mathbb{C}) \arrow[u, hook] \end{tikzcd}$$

$\varphi(h)$'s span $M_{n_i}(\mathbb{C})$

$\Omega_{\varphi,K}$ commutes w/ these $\Rightarrow$

$\Omega_{\varphi,K} \in Z(M_{n_i}(\mathbb{C}))$

$\Rightarrow \Omega_{\varphi,K} = \omega_{\varphi,K} I$

$$\operatorname{tr}(\Omega_{\varphi,K}) = \operatorname{tr} \sum_{g\in K} \varphi(g) = \sum_{g\in K} \operatorname{tr}\varphi(g)$$

$$= \sum_{g\in K} \chi(g) = |K|\,\chi(g_j)$$

$g_j \in K_j = K$

$\varphi_i \, , \chi_i \qquad K_j \ni g_j$

$$\Omega_{\varphi, K} \leftrightarrow \Omega_{i,j} = \omega_{ij} I$$

$$\operatorname{tr}(\Omega_{ij}) = |K_j| \chi_i(g_j)$$

$$\operatorname{tr}(\omega_{ij} I) = \omega_{ij} \operatorname{tr}(I) = \omega_{ij} \chi_i(1)$$

$$\boxed{\omega_{ij} = \frac{|K_j| \chi_i(g_j)}{\chi_i(1)}} \in \mathbb{C}$$

$\omega_{ij} \, \omega_{ik} \in \mathbb{Z}$-lin comb of ω_{il}'s

(form a Ring)

$$\Rightarrow \left\{ \sum a_j \omega_{ij} \;\middle|\; a_j \in \mathbb{Z} \right\} = R_i \subset \mathbb{C}$$

non-unital subring.

$$\Omega_{\varphi,K}\,\Omega_{\varphi,K'} = \Big(\sum_{g\in K}\varphi(g)\Big)\Big(\sum_{h\in K'}\varphi(h)\Big)$$

$$= \sum_{\substack{g\in K\\ h\in K'}}\varphi(gh) = \sum_{k\in G}\Big(\sum_{\substack{g\in K\\ h\in K'\\ \text{s.t. } gh=k}}\varphi(k)\Big)$$

$$= \sum_{k\in G}\#\{g\in K, h\in K' \mid gh=k\}\,\varphi(k)$$

to show this is a lin. comb. of $\Omega_{\varphi,K''}$'s

need to show, if k_1, k_2 are conjugate

then coeff of $\varphi(k_1)$ = coeff of $\varphi(k_2)$

$l k_1 l^{-1} = k_2$

$$\begin{tikzcd}
\{g\in K, h\in K' \mid gh=k_1\} \arrow[d, shift right, "{g,h \,\mapsto\, lgl^{-1},\, lhl^{-1}}"'] \\
\{g\in K, h\in K' \mid gh=k_2\} \arrow[u, shift right, "l^{-1}_\,l"']
\end{tikzcd}$$

Δ.

Cor $\mathbb{Z} + \sum_j \omega_{i,j}\mathbb{Z} \subset \mathbb{C}$ is a subring.

finitely generated / $\mathbb{Z}$.

$\Rightarrow$ elements are algebraic integers

Def/Lem The following are equivalent for $\alpha \in \mathbb{C}$

- α is the root of some monic poly in $\mathbb{Z}[X]$
- $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a finite field ext., and $\min(\alpha) \in \mathbb{Z}[X]$
- $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module
- $\mathbb{Z}[\alpha]$ is contained in a f.g. $\mathbb{Z}$-mod.

Such an α is called an algebraic integer.

Lem algebraic integers form a ring.

Lem if $\alpha \in \mathbb{Q}$ is an algebraic integer $\Rightarrow \alpha \in \mathbb{Z}$.

Pf: $\min(\alpha) = X - \alpha \in \mathbb{Z}[X] \Rightarrow \alpha \in \mathbb{Z}$.

Lem if $g \in G$, χ a character $\Rightarrow \chi(g)$ an alg. integer.

Pf: WLOG, can assume $G = \langle g \rangle$ – in particular G abelian.

can also assume $\chi \leftrightarrow \varphi$ irrep.

$$\varphi = \varphi_1 \oplus \cdots \oplus \varphi_e \Rightarrow \operatorname{tr} \varphi = \sum \operatorname{tr} \varphi_i$$

$$\chi = \sum \chi_i$$

$$\chi(g) = \sum \chi_i(g)$$

e.t.s. $\chi_i(g)$ an alg int.

G Ab $\Rightarrow \mathbb{C}G$ Ab $\quad \mathbb{C}G = \times M_{n_i}(\mathbb{C})$

$$\Rightarrow n_i = 1 \text{ all } i$$

$\Rightarrow \varphi$ is 1-dim'l

$$\varphi : G \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^*$$

$$g \longmapsto \text{root of unity}$$

$$g^m \mapsto 1 \qquad m = \operatorname{ord}(g)$$

$$\varphi(g) \text{ root of } X^n - 1$$

$$\operatorname{tr} \varphi(g) = \chi(g). \qquad \square$$

Prop

$\chi_i(1) \mid |G|$ for each $\overset{\text{char of}}{\text{irreps}}$ $\chi_1, \dots, \chi_r$.

Pf

$$\underset{\overset{\uparrow}{\mathbb{Q}}}{\frac{|G|}{\chi_i(1)}} = \frac{|G|}{\chi_i(1)} \langle \chi_i, \chi_i \rangle$$

$$= \frac{|G|}{\chi_i(1)} \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_i(g)}$$

$$= \sum_{g \in G} \frac{\chi_i(g)}{\chi_i(1)} \overline{\chi_i(g)}$$

$$= \sum_{j=1}^{r} \sum_{g \in K_j} \frac{\chi_i(g)}{\chi_i(1)} \overline{\chi_i(g)}$$

$g_j \in K_j$

$$= \sum_{j=1}^{r} \frac{\chi_i(g_j)|K_j|}{\chi_i(1)} \overline{\chi_i(g)}$$

$$= \sum_{j=1}^{r} \omega_{i,j} \overline{\chi_i(g)}$$ is an alg integer!

in $\mathbb{Q} \Rightarrow$ in $\mathbb{Z}$

so $\dfrac{|G|}{\chi_i(1)} \in \mathbb{Z}$ $\square$.

Lem If φ is a ~~rep~~ irrep w/ character χ, K a conj class and $(|K|, \chi(1)) = 1 \Rightarrow$ for $g \in K$

$\chi(g) = 0$ or $\varphi(g)$ is a scalar matrix.

Pf: write $s|K| + t\chi(1) = 1$ mult by $\dfrac{\chi(g)}{\chi(1)}$

$$\rightsquigarrow s \underbrace{\frac{|K|\chi(g)}{\chi(1)}}_{\substack{\omega_{\chi,K} \\ \text{alg int}}} + t \underbrace{\chi(g)}_{\substack{\text{alg} \\ \text{int}}} = \frac{\chi(g)}{\chi(1)} \in \text{alg int}$$

$$\varphi(g) = \underbrace{\begin{bmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{bmatrix}}_{T}$$

$\varepsilon_i^m = 1 \quad m = o(g)$

$\varepsilon_i \in$ some Gal ext $E/\mathbb{Q}$

$|\mathrm{Gal}(E/\mathbb{Q})| = N$

$$\operatorname{tr} \sum_{\sigma \in \mathrm{Gal}(E/\mathbb{Q})} \sigma(T) = \sum \mathrm{conj}\, \chi(g)$$

$$\sum_{\sigma \in Gal} \sigma(\chi(g)) = N \cdot n \text{ (mth roots of 1)}$$

$$z = \frac{\sum \varepsilon_i}{n} \quad \text{an alg integer} \quad |z| \leq 1$$

$$\left| \prod_{\sigma \in Gal} \sigma(z) \right| \leq 1$$

alg int. fixed by Gal $\Rightarrow \mathbb{Q} \Rightarrow \mathbb{Z}$

$$\Rightarrow z = 0 \text{ or } |z| = 1 \qquad z = \frac{\chi(g)}{\chi(1)}$$

$\Downarrow$ $\chi(g) = 0$

$\Downarrow$ $\left| \sum \varepsilon_i \right| \leq \sum |\varepsilon_i|$

w/ equality if all pos mult. of same vector.

$\Rightarrow$ all equal

$\Rightarrow \varphi(g)$ diagonal. $\square$

Lem (final)

Suppose G is finite, K a conj class $K \neq \{e\}$

$|K| = p^c$ $c \geq 0$ then G cannot be simple, nonabelian.

Pf: if $c = 0 \Rightarrow |K| = 1$, and if $g \in K$

$\Rightarrow g \in Z(G) \Rightarrow G$ not simple & ...

Let χ be regular rep. Consider

$$\langle \chi, \chi_i \rangle = \frac{1}{|G|} \sum \chi_i(g) \overline{\chi(g)}$$

$$= \frac{1}{|G|} \chi_i(1) \overline{\chi(1)} = \chi_i(1)$$

$$\chi = \sum \chi_i \langle \chi, \chi_i \rangle = \sum \chi_i \chi_i(1)$$

choose $g \in K$

$$\chi(g) = 0 = \underset{\underset{1}{\shortparallel}}{\chi_1(g)} + \sum_{i=2}^{n} \chi_i(1) \chi_i(g)$$

χ_1 = trivial 1 dim'l rep

consider i's s.t. $\chi_i(g) \neq 0$

if $p \mid \chi_i(1)$ s.t. $\chi_i(g) \neq 0$

$$\Rightarrow 0 = 1 + \sum_{\substack{\text{nice} \\ i\text{'s}}} \chi_i(1)\chi_i(g)$$

$$= 1 + p \sum d_i \underbrace{\chi_i(g)}_{\text{alg ints}}$$

($d_i \in \mathbb{Z}$)

$\Rightarrow -\frac{1}{p}$ alg int $\frac{1}{2}$.

$\Rightarrow \exists$ i s.t. $p \nmid \chi_i(1)$, $\chi_i(g) \neq 0$

$|K| = p^e$, $p \nmid \chi_i(1)$

prev lemma $\varphi_i(g) = 0$ (trace) or scalar

$\chi_i(g) \neq 0 \Rightarrow$ scalar

note if G simple $\varphi_i : G \to GL_{n_i}(\mathbb{C})$

is injective.

$\Rightarrow \varphi g$ commutes w/ image

$\Rightarrow g \in Z(G)$

$\Rightarrow$ not simple D.