$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$(-1)^2 = 1 \qquad i^2 = j^2 = k^2 = -1 \qquad (-1)\,i = -i$$

$$ij = -ji = k$$

$$\begin{tikzcd}[column sep=small]
 & Q_8 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle i \rangle \arrow[dr, dash] & \langle j \rangle \arrow[d, dash] & \langle k \rangle \arrow[dl, dash] \\
 & \langle -1 \rangle \arrow[d, dash] & \\
 & \langle 1 \rangle &
\end{tikzcd}$$

not characteristic
all normal

characteristic.

Today's topic: Group actions

Def Let G be a group, X a set.

A (left) action of G on X is a tuple (G, X, a)

$$a : G \times X \longrightarrow X$$
$$(g, s) \longmapsto g \cdot s = a(g, s)$$

such that $\forall g, h \in G,\ x \in X$, $\quad g(hx) = (gh)x$

$$ex = x$$

Altrnate language X is a "G-Set"

$$X = (G, X, a)$$

A homomorphism of G-Sets is a map of sets

$$f : X \longrightarrow Y \text{ s.t.}$$

$$f(g \cdot x) = g \cdot f(x)$$

isomorphisms, automorphisms, epimorphisms, monomorphisms

(surjective) (injective)

sub-actions / sub-G-sets.

Def 2 | Let G be a group, X a set, a G-set is a homomorphism $G \longrightarrow S_X = \{ \varphi : X \to X \mid \varphi \text{ is a bijection} \}$

Given $\alpha : G \longrightarrow S_X$, $\beta : G \longrightarrow S_Y$, a hom of G Set's from $X \to Y$ is $f : X \to Y$ s.t.

$$f(\alpha(g)(x)) = \beta(g)(f(x))$$

<u>Note</u> Categories of G-sets & G-Set's are isomorphic categories.

$$G\text{-Set } (G, X, a) \longrightarrow G\text{-Set}'$$

$$G \longrightarrow S_X$$

$$g \longmapsto (x \longmapsto g\cdot x)$$

$$g\cdot x = \alpha(g)(x) \longleftarrow G \xrightarrow{\alpha} S_X$$

<u>Ex:</u> "Regular action"

- G acts on itself via left mult. $G = X$

  $g \cdot h = gh$

  $$G \longrightarrow S_G$$

  inj (Cayley's thm)

- Trivial action on a set X

  $g \cdot x = x$

  $$G \longrightarrow S_X$$
  $$g \longmapsto e$$

- Coset action

$$G \text{ acts on } X = G/H = \{gH \mid g \in G\}$$

$$g \cdot (g'H) = gg'H$$

$$G \to S_{G/H}$$

- Conjugation action

$$G \text{ acts on } G \text{ via } g \cdot h = ghg^{-1} = \mathrm{inn}_g(h)$$

$$G \to S_G$$

- Conj. action on subgroups

$$X = \mathrm{Sub}(G) = \{H < G\}$$

$$g \cdot H = gHg^{-1}$$

- Automorphism action on G

$$\mathrm{Aut}\, G \text{ acts on } G$$

$$\varphi \cdot g = \varphi(g)$$

• Aut action on subgps

$$\mathrm{Aut}\, G \circlearrowright \mathrm{Sub}(G)$$

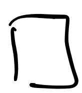
"acts on"

$$\varphi \cdot H = \varphi(H)$$

Specific actions:

S_n acts on $\{1, \dots, n\}$

D_{2n} acts on $\{1, \dots, n\}$ vertices on n-gon

D_8  also acts on 2 diagonals.

$\mathbb{Z}/n\mathbb{Z} = C_n$ acts on $\{1, \dots, n\}$

$GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$

__Def__ If G acts on X, $x \in X$,
define $\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\}$

__note:__ $\mathrm{Stab}_G(x) < G$

Def If X is a G-set, define kernel of the action $\ker(X) = \ker(G \to S_X)$

$$\ker X \trianglelefteq G.$$

Def If G acts on X, $x \in X$, we define the orbit of x, written $Gx = \{gx \mid g \in G\}$

Note Actions give eq. relations

$$x \sim y \text{ if } y = gx \text{ some } g \in G$$

orbits = eq. classes.

also get groupoids $\quad x \xrightarrow{g} y \quad gx = y$

Prop Suppose $G \circlearrowright X$ and $gx = y$ then $\mathrm{Stab}_G(y) = g\,\mathrm{Stab}_G(x)\,g^{-1}$

Pf: $\supseteq$ clear. $s \in \mathrm{Stab}_G(x) < G$

$$(gsg^{-1}).y = (gsg^{-1})gx = gsx = gx = y$$

action $(gsg^{-1}g).x$

$\subseteq \quad gx=y \Leftrightarrow g^{-1}y=x$

prev $\Rightarrow g^{-1}\mathrm{Stab}_G(y)(g^{-1})^{-1} \leq \mathrm{Stab}_G(x)$

$\mathrm{Stab}_G(y) \leq g\,\mathrm{Stab}_G(x)g^{-1}$

$\square$.

Def $\mathrm{Tran}_G(x,y) = \{g \in G \mid gx=y\}$

Prop If $y=gx$ then

$$\mathrm{Tran}_G(x,y) = g\,\mathrm{Stab}_G(x)$$

Pf: $\supseteq$ clear

$\subseteq$ note if $t \in \mathrm{Trans}_G(x,y) \Rightarrow g^{-1}t \in \mathrm{Stab}_G(x)$

$\Downarrow$

$t \in g\,\mathrm{Stab}_G(x)$. $\square$.

Cor $|\mathrm{Trans}_G(x,y)| = |\mathrm{Stab}_G(x)|$

if $y \in Gx$.

Theorem $\boxed{\text{Orbit-Stabilizer theorem}}$

aka. "most important theorem in math"

Suppose X is a G-set, $x \in X$, then

$$|Gx| = [G : \mathrm{Stab}_G(x)] = \frac{|G|}{|\mathrm{Stab}_G(x)|}$$

Pf: $\exists$ surjective map

$$G \xrightarrow{\varphi} Gx$$

$$g \longmapsto gx$$

For $y \in Gx$, $\varphi^{-1}(y) = \mathrm{Trans}_G(x,y)$

$$|\varphi^{-1}(y)| = |\mathrm{Stab}_G(x)|$$

because of the meaning of the word multiplication, $|G| = |\mathrm{Stab}_G(x)|\,|Gx|$ $\square$.

$G \circlearrowright G/H$ left mult.

single orbit!

$$\mathrm{Stab}_G(H) = \{g \in G \mid gH = H\} = H$$

thm $\Rightarrow$ $|G/H| = \dfrac{|G|}{|H|}$ Lagrange

- $G \circlearrowright \mathrm{Sub}(G)$ conj

  $H < G$ $\quad \mathrm{Stab}_G(H) = \{g \in G \mid gHg^{-1} = H\}$

  $\quad\quad\quad ///$

  $\quad\quad\quad N_G(H)$ "normalizer"

  largest subgp containing K H s.t. $H \triangleleft K$

Prop Suppose $H < G$ s.t. $\left(([G:H]-1)!, |H| \right) = 1$

then $H \triangleleft G$.

Pf: Consider $G \circlearrowright G/H$ by left mult.

Get $G \xrightarrow{\varphi} S_{G/H}$, and $\ker\varphi \subset \mathrm{Stab}_G H = H$

by 1st iso $G/\ker\varphi < S_{G/H} \Rightarrow$

$$|G/\ker\varphi| \,\Big|\, |S_{G/H}| = [G:H]! = [G:H]\cdot m$$

$m = ([G:H]-1)!$

$$\Rightarrow \frac{|G|}{|\ker\varphi|} \,\Big|\, \frac{|G|}{|H|}\cdot m \qquad \text{mult by } |H|$$

$$|G|\cdot\frac{|H|}{|\ker\varphi|} \,\Big|\, |G|\cdot m$$

integer since $\ker\varphi < H$

$$\Rightarrow \frac{|H|}{|\ker\varphi|} \,\Big|\, m$$

$$[H:\ker\varphi] \mid m$$

assumption $|H|$ prime to m.

$$\Rightarrow [H:\ker\varphi]=1 \Rightarrow \ker\varphi = H \trianglelefteq G.$$

$\square$.

<u>Def</u> $C_G(A) = \{g\in G \mid ga=ag \text{ all } a\in A\}$

$A\subset G$ subset $\qquad gag^{-1}=a$

$$= \bigcap_{a\in A} \mathrm{Stab}_G(a) < G$$

w/r/t conj action.

Def $Z(G) = C_G(G)$

$= \ker(\text{inn} : G \to S_G)$

$Z(G) \trianglelefteq G$

Def g_1 & g_2 are conjugate if $\exists h \in G$ s.t.

$hg_1h^{-1} = g_2$

(i.e. same orbit w/ conj action).

Conjugacy classes are <u>magical</u>.

$G = \bigsqcup$ conj classes $\bigsqcup$ = "disjoint union"

$= \bigsqcup$ orbits under conj. $\bigcup$ = "union"

choose representatives of conj classes.

$z_1, \ldots, z_r, g_1, \ldots, g_m$ where $|\text{conj class of } z_i| = 1$

$|\text{conj classes of } g_i| > 1$

$Z(G) = \{z_1, \ldots, z_r\}$

$$|\text{conj class of } g_i| = \frac{|G|}{|\mathrm{Stab}_G(g_i)|} = [G : C_G(g_i)]$$

<u>Theorem</u> "The class equation" (Key to power)

$$|G| = |Z(G)| + \sum_{i=1}^{m} [G : C_G(g_i)]$$

$g_1, \dots, g_m$ reps of nontrivial conj classes