Putting groups together.

Suppose we have a gp G & normal subgp N, quotient $G/N = \overline{G}$.

<u>Q</u>: Given N, G/N, How to construct G?

Explicit construction:

$$G = \bigcup Ng = \bigsqcup_{\bar{g} \in G/N} N s(\bar{g})$$

where $s: G/N \to G$ is a set map section.

by def, $s(\bar{g}) s(\bar{h}) \in N s(\bar{g}\bar{h})$

so there exists some $\nu(\bar{g}, \bar{h})$ (depending on s) s.t.

$$s(\bar{g}) s(\bar{h}) = \nu(\bar{g}, \bar{h}) s(\bar{g}\bar{h})$$

- So, to describe mult, need to know "correction factors" $\nu(\bar{g},\bar{h})$

these are the correctors to s being a hom.

Also, mult. rule:

$$n\, s(\bar{g})\, m\, s(\bar{h}) = n\, s(\bar{g})\, m\, s(\bar{g})^{-1} s(\bar{g}) s(\bar{h})$$
$$= n\ \mathrm{inn}_{s(\bar{g})}(m)\ \nu(\bar{g},\bar{h})\ s(\bar{g}\bar{h})$$

So, need to know the inner action of $s(\bar{g})$ on N.

ν holds most of the complications, can't be arbitrary, but need to satisfy associativity

$$s(\bar{g})\, s(\bar{h})\, n = \nu(\bar{g},\bar{h})\, s(\overline{gh})\, n$$
$$= \nu(\bar{g},\bar{h})\ \overline{gh}(n)\ s(\overline{gh})$$

$\parallel$

$s(\bar{g})\, \bar{h}(n)\, s(\bar{h})$

$\parallel$

$$\bar{g}\,\bar{h}(n)\, s(\bar{g})\, s(\bar{h}) = \bar{g}\,\bar{h}(n)\,\nu(\bar{g},\bar{h})\, s(\bar{g}\bar{h})$$

so $\boxed{\nu(\bar g,\bar h)\,\overline{gh}(n) = \bar g\,\bar h(n)\,\nu(\bar g,\bar h)}$

$$s(\bar g)s(\bar h)s(\bar k) = s(\bar g)\,\nu(\bar h,\bar k)\,s(\bar h\bar k)$$

$$\nu(\bar g,\bar h)\,s(\bar g\bar h)\,s(\bar k) \qquad \bar g(\nu(\bar h,\bar k))\,\nu(\bar g,\bar h\bar k)$$

$$= \nu(\bar g,\bar h)\,\nu(\bar g\bar h,\bar k)\,s(\;) \qquad s(\;)$$

so $\boxed{\nu(\bar g,\bar h)\,\nu(\bar g\bar h,\bar k) = \bar g(\nu(\bar h,\bar k))\,\nu(\bar g,\bar h\bar k)}$

<u>Complicating factors</u>

$s(\bar g)s(\bar h) \neq s(\bar g\bar h)$

leads to the ν's & also the irritating fact that

$$\mathrm{inn}_{s(\bar g)s(\bar h)} \neq \mathrm{inn}_{s(\bar g\bar h)}$$

which leads to a complicated condition on ν's.

Fixes (commonly used)

- Pick an s which is a homomorphism
  — get no ν's, group structure is determined by inner action of G/N on N.
  "semidirect products"
- Assume $N \subset Z(G)$
  — inner actions are trivial, still need s, but ν identity is simpler.
  "central extensions"

Both are very useful

central extensions can classify p-grp's,
ν rules can be enumerated
i.e. can classify ν corresp. to any possible s's.

computationally feasible. e.g. classify gps order 16 or 125 etc.

<u>Next</u>: semidirect products

excellent tool for gps w/ few prime factors.

$$\begin{tikzcd} N \arrow[r, hook] & G \arrow[r] & G/N \arrow[d, phantom, "\cong"] \\ & & H \arrow[ul, bend left] \end{tikzcd}$$

every thing in G of form nh uniquely

so $G = NH$.

group elements correspond to pairs.

mult: $(nh)(n'h') = n(hn'h^{-1})hh'$

<u>Def</u>: Suppose we have gps N, H and $\varphi: H \to \mathrm{Aut}\,N$, $h \mapsto \varphi_h$

we define $N \rtimes_\varphi H$ or $N \rtimes H$ to be pairs

(n,h) w mult. $\quad (n,h)(n',h') = (n\,\varphi_h(n'), hh')$

Thm TFAE $\quad$ Given G, H, N

1) $G \cong N' \rtimes H'$

2) $\exists N \trianglelefteq G$, $N \cong N'$ and $G/N \cong H'$

and $s : G/N \to G$ a hom. s.t. $\begin{tikzcd} G/N \arrow[r, "s"] \arrow[rr, bend right, "id"'] & G \arrow[r, "\pi"] & G/N \end{tikzcd}$

3) $\exists N \trianglelefteq G$, $H < G$ w/ $N \cong N'$, $H \cong H'$

s.t. $NH = G$ and $N \cap H = (e)$

Pf: $\quad 1 \Rightarrow 2$ ✓

$2 \Rightarrow 3$ recall $|NH| = \dfrac{|N||H|}{|N\cap H|}$ and $|N||H| = |G|$ by 2.

✓

$3 \Rightarrow 1$ ✓