Last week:

learned how to take groups apart

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ N & & G/N \end{tikzcd}$$

This week:

Learn to put groups back together.

Q: Given N, G/N, find G

$$G = \bigcup Ng$$

$G \xrightarrow{\pi} G/N$ choose $s: G/N \to G$ set map

such that $\pi(s(\bar{g})) = \bar{g}$

$$N s(\bar{g}) = \bar{g}$$

$$G = \bigsqcup_{\bar{g} \in G/N} N s(\bar{g})$$

unique expression

$$\Longrightarrow G = \{ n s(\bar{g}) \mid n \in N, \bar{g} \in G/N \}$$

$n s(\bar{g})\, m\, s(\bar{h}) = p\, s(\bar{k})$ some $p, \bar{k}$

2 ingredients: $s(\bar{g}) m = \underline{\quad}$?

$s(\bar{g}) s(\bar{h}) = \underline{\quad}$?

$$s(\bar{g}) m = \underbrace{s(\bar{g}) m s(\bar{g})^{-1}}_{\in N} s(\bar{g})$$

$$= inn_{s(\bar{g})}(m)\, s(\bar{g})$$

need extra information of how $s(\bar{g})$ acts on N via inner aut.

$s(\bar{g}) s(\bar{h}) \in N s(\bar{g}\bar{h}) = \bar{g}\bar{h}$

$$\pi\left(s(\bar{g}) s(\bar{h})\right) = \pi s(\bar{g})\, \pi s(\bar{h})$$

$$= \bar{g}\bar{h}$$

$$s(\bar{g}) s(\bar{h}) = \nu(\bar{g},\bar{h})\, s(\bar{g}\bar{h})$$

Ingredient #2 mysterious fcn $\nu : G/N \times G/N \to N$

Given N, $\bar{G}$, $\varphi : \bar{G} \to \mathrm{Aut}(N)$ setmap

$\bar{G} = G/N$, $\nu : \bar{G} \times \bar{G} \to N$

when can we find G, $N \trianglelefteq G$, $G/N = \bar{G}$

$s : G/N \to G$

s.t. $\varphi(\bar{g})(n) = \mathrm{inn}_{s(\bar{g})}(n)$

$$s(\bar{g})s(\bar{h}) = \nu(\bar{g},\bar{h})\, s(\bar{g}\bar{h})$$

<u>Need to check:</u>

$$\big(s(\bar{g})s(\bar{h})\big)s(\bar{k}) = s(\bar{g})\big(s(\bar{h})s(\bar{k})\big)$$

$$\big(s(\bar{g})s(\bar{h})\big)n = s(\bar{g})\big(s(\bar{h})n\big)$$

$$\nu(\bar{g},\bar{h})\, s(\bar{g}\bar{h})\, s(\bar{k})$$

$$= \nu(\bar{g},\bar{h})\, \nu(\bar{g}\bar{h},\bar{k})\, s(\bar{g}\bar{h}\bar{k})$$

$$s(\bar{g})\, \nu(\bar{h},\bar{k})\, s(\bar{h}\bar{k})$$

$$= s(\bar{g})\nu(\bar{h},\bar{k})\, s(\bar{g})^{-1} s(\bar{g})\, s(\bar{h}\bar{k})$$

$= \mathrm{inn}_{s(\bar{g})}(\nu(\bar{h},\bar{k}))\,\nu(\bar{g},\bar{h}\bar{k})\,s(\bar{g}\bar{h}\bar{k})$

Valid ν's must satisfy:

$$\nu(\bar{g},\bar{h})\,\nu(\bar{g}\bar{h},\bar{k}) = \mathrm{inn}_{s(\bar{g})}(\nu(\bar{h},\bar{k}))\,\nu(\bar{g},\bar{h}\bar{k})$$

i.e. abstract data $\bar{G}, N, \varphi: \bar{G} \to \mathrm{Aut}\,N$

$\nu: \bar{G}\times\bar{G} \to N$ must satisfy

$$\nu(\bar{g},\bar{h})\,\nu(\bar{g}\bar{h},\bar{k}) = \varphi(\bar{g})(\nu(\bar{h},\bar{k}))\,\nu(\bar{g},\bar{h}\bar{k})$$

"non-Abelian 2-cocycle condition"

and another identity from $s(\bar{g})s(\bar{h})n$

$(s(\bar{g})s(\bar{h}))n$

$s(\bar{g})(s(\bar{h})n)$

$$\rightsquigarrow \nu(\bar{g},\bar{h})\,\varphi(\bar{g}\bar{h})(n) = \varphi(\bar{g})(\varphi(\bar{h})(n))\,\nu(\bar{g},\bar{h})$$

?

Complicating factors

- Failure for s being a homomorphism

if s is hom, $\varphi: \bar{G} \to \mathrm{Aut}\,N$ hom, $\nu =$ trivial "semidirect product"

- inner action φ = trivial

cocycle condition is main nontriv. thing

if $N \subset Z(G)$ "central extensions"

→ are "computable"

<u>Language:</u> if given G, $N \triangleleft G$, $\bar{G} = G/N$

we say G is an extension of $\bar{G}$ by N.

Reasonable when $N \subset Z(G)$

Recall: if G is a p-group ($|G| = p^n$)

then $Z(G) \neq (e) \Rightarrow G$ is always a nontrivial central ext. of a smaller p-gp

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Semidirect Products

$$\begin{tikzcd} N \arrow[r, hook, "\triangleleft"] & G \arrow[r, "\pi"] & G/N \arrow[l, bend left, "s"] \\ & & H \arrow[ul, bend left] \end{tikzcd}$$

$G/N \cong H$

i.e. have G, $N \triangleleft G$, $H < G$, have $s : G/N \xrightarrow{\sim} H$

such that $\pi s = id_{G/N}$

as before, every elmt of G has the form nh
$n \in N, h \in H$

i.e. $G = NH$

<u>multi</u>: $(nh)(n'h') = n \underbrace{hn'h^{-1}}_{inn_h(n')} hh'$

<u>Def</u> Given groups N, H, $\varphi : H \to \operatorname{Aut} N$ homomorphism
$h \mapsto \varphi_h$

we define $N \rtimes_\varphi H = N \rtimes H$ to be pairs (n,h)
$n \in N, h \in H$

w/ mult. rule $(n,h)(n',h') = (n\,\varphi_h(n'), hh')$

<u>check</u>: this is a group.

<u>Def</u> Given G, $N \triangleleft G$, $H < G$ such that
$NH = G$, $N \cap H = (e)$, we say that
G is a internal semidirect product of N & H
$G = N \rtimes H$

Remark: If $G = N \dot{\rtimes} H$ then $G \cong N \rtimes_\varphi H$

$$\varphi : H \to \operatorname{Aut} N$$
$$\varphi_h(n) = hnh^{-1}$$

Pf: $G = N \dot{\rtimes} H \longleftrightarrow N \rtimes_\varphi H$

$$nh \longleftrightarrow (n,h)$$

$\ldots$

$(nh)(n'h')$

$= nhn'h' = nhn'h^{-1}hh'$

unique since $N \cap H = (e)$

$= (n\, \mathrm{inn}_h(n'), hh')$

Remark: if $N, H \triangleleft G$, $NH = G$, $N \cap H = (e)$

then N & H commute and $G = N \times H$

Lemma If $N, H < G$, $N \cap H$, $H < N_G(N)$

$N < N_G(H)$

then $nh = hn$ $\forall n \in N, h \in H$.

Pf: $nhn^{-1}h^{-1} = (nhn^{-1})h^{-1} \in HH \subset H$

$\shortparallel$

$n(hn^{-1}h^{-1}) \in NN = N$

$\Rightarrow nhn^{-1}h^{-1} = e$

$nh = hn$ $\square$

in this case $G \simeq N \rtimes H = N \rtimes_1 H$

$= N \times H$

Ex: $|G| = 15$

$n_5 \equiv 1 \ (5) \quad n_5 | 3 \quad n_5 = 1$

$n_3 \equiv 1 \ (3) \quad n_3 | 5 \quad n_3 = 1$

$P_3, P_5 \lhd G \quad P_3 P_5 = G$

$P_3 \cap P_5 = (e)$

$G \simeq P_3 \times P_5 = \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

$|G| = 14 \qquad n_7 = 1 \qquad n_2 = ?$

$P_7 P_2 = G \qquad P_7 \cap P_2 = (e)$

$G \simeq P_7 \rtimes_\varphi P_2 \qquad P_2 \xrightarrow{\varphi} \operatorname{Aut} P_7$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad U_7 = \mathbb{Z}/6\mathbb{Z}$

$G \simeq \mathbb{Z}_{14}$ or D_{14} $\qquad\qquad\qquad \shortparallel$

and that's it. $\quad (\mathbb{Z}/7\mathbb{Z})^*$