An accounting of structures

$$\begin{tikzcd} & \text{subgroups} \\ G \arrow[ur] \arrow[r] \arrow[dr] & p\text{-subgroups} \\ & p\text{-Sylow subgroups} \end{tikzcd}$$

Lattices

sets.

sets w/ G action

Recall: partially ordered set $(S, \leq)$

set

binary relation

$a \leq a$ reflexive

$a \leq b,\ b \leq c \Rightarrow a \leq c$ transitive

$a \leq b,\ b \leq a \Rightarrow a = b$ anti-symmetric

Lattice: $\forall\, a,b \in S\ \exists\ c, c' \in S$ s.t. $c \leq a,b$ and $d \leq a,b \Leftrightarrow d \leq c$ ← greatest lower bound

least upper bound. → $c' \geq a,b$ and $d' \geq a,b \Leftrightarrow d' \geq c'$

<u>Recall</u> A monoid-oid is a pair of sets

C_0 , C_1 , and set maps $s, t: C_1 \to C_0$
"source & target"

partially defined composition

$f \circ g$ $\quad f, g \in C_1$ when $s(f) = t(g)$

, identity arrows, associativity, etc.

<u>Def</u> A category $\mathcal{C}$ consists of a set (or class) of objects $ob(\mathcal{C})$ $= C_0$ and for $a, b \in ob(\mathcal{C})$

a set of morphisms $Hom_{\mathcal{C}}(a,b)$ $= \{f \in C_1 \mid s(f) = a,\ t(f) = b\}$

together with distinguished morphisms $1_a \in Hom_{\mathcal{C}}(a,a)$ all a,

and composition rule $Hom(B,C) \times Hom(A,B) \to Hom(A,C)$

such that

- $(f \circ g) \circ h = f \circ (g \circ h)$
- $f \cdot 1_a = f = 1_b f$ if $f \in Hom(a,b)$

<u>Def</u> A functor $F: \mathcal{C} \to \mathcal{D}$ between cats $\mathcal{C}, \mathcal{D}$ is a map of objects $F: ob(\mathcal{C}) \to ob(\mathcal{D})$ and for each $a, b \in ob(\mathcal{C})$, a map of morphisms

$$F: Hom_{\mathcal{C}}(a,b) \to Hom_{\mathcal{D}}(Fa, Fb)$$

such that

- $F(1_a) = 1_{F(a)}$ and
- $F(fg) = F(f)F(g)$

<u>examples</u>

$$\text{Groups} \xrightarrow{\text{subgp}\ "F"} \text{Sets}$$

$$G \longmapsto \text{Subgp}(G) = \{H < G\}$$

$$(\varphi: G \to G') \longmapsto [F\varphi : H \longmapsto \varphi(H)]$$

$$id: G \to G \longmapsto Fid: H \mapsto H$$

$$= id_{\text{Subgp}(G)}$$

$G \xrightarrow{\varphi} G' \xrightarrow{\psi} G''$, want

$$F(\psi\varphi) = F(\psi)F(\varphi)$$

$$\psi\varphi(H)$$

$$(F\varphi)(H) = \varphi(H)$$

$$(F\psi)(\varphi(H)) = \psi(\varphi(H)) = \psi\varphi(H)$$

$$\text{Groups} \longrightarrow \text{Po sets/ (Lattices)}$$

= objects are po sets
morphisms are
set maps
preserving order.

$$G \longmapsto \text{Subgp}(G)$$

$$\text{SurjGroup} \longrightarrow \text{Ab Groups}$$

$$G \longrightarrow Z(G)$$

$$G \to G'$$

objects groups, morphisms = surjective homs.

$$G \longrightarrow G$$

$$\text{SurjGps} \longrightarrow \text{Gps}$$

$$\longrightarrow \text{Gps}$$

$$G \longrightarrow Z(G)$$

Def: Given functors $F, G : \mathcal{C} \to \mathcal{D}$, we define a natural transformation $\alpha : F \Rightarrow G$ to be a rule which associates to each $c \in \mathcal{C}$ a morphism $\alpha_c \in \mathrm{Hom}_{\mathcal{D}}(F(c), G(c))$

$\alpha_c : F(c) \to G(c)$ such that

whenever $f \in \mathrm{Hom}_{\mathcal{C}}(c, c')$, we have a comm. digram (in $\mathcal{D}$)

$$\begin{tikzcd} F(c) \arrow[r, "Ff"] \arrow[d, "\alpha_c"'] & F(c') \arrow[d, "\alpha_{c'}"] \\ G(c) \arrow[r, "Gf"'] & G(c') \end{tikzcd}$$

Ex: $$\begin{tikzcd} \mathrm{SurjGp} \arrow[r, "\mathrm{self}"] \arrow[dr, "Z"'] & \mathrm{Gp} \\ & \mathrm{Gp} \end{tikzcd}$$

get a natural transformation $\alpha : Z \Rightarrow \mathrm{self}$

$G \longmapsto [Z(G) \overset{\alpha_G}{\hookrightarrow} G]$

Central series

"Ascending central series" (upper central series)

Def An asc. central series for a group G is a sequence of subgroups

$$(e) = Z'_0 \subset Z'_1 \subset \dots \subset Z'_n = G$$

such that $Z'_i \lhd G$ and

$$Z'_{i+1}/Z'_i \subseteq Z(G/Z'_i)$$

If this exists, we say that G is <u>nilpotent</u>.

Def The ascending (upper) central series for G is the sequence $Z_0 = (e)$ $\quad Z_{i+1} = Z(G/Z_i)$

Claim: G is nilpotent $\Longleftrightarrow$ the ascending central series terminates at G

Pf: $Z'_i \subset Z_i$ all Z'_i s.

If G is nilpotent, the length n of the series (asc. central) is called the nilpotentcy class.

<u>Theorem</u> The following are equivalent for a finite group G

1. • G is nilpotent
2. • $G = P_1 \dot{\times} \cdots \dot{\times} P_m$ where P_i are the Sylow subgroups
3. • All Sylow subgroups are normal
4. • If $H \underset{\neq}{<} G$ then $N_G(H) \supsetneq H$

<u>Pf.</u> $1 \Rightarrow 4$

G nilpotent, $H \underset{\neq}{\leq} G$. consider $Z(G) \neq (e)$.

if $Z(G) \not\subseteq H$, then $g \in Z(G) \setminus H$, $g \in N_G(H) \setminus H$

if $Z(G) \subseteq H$, then consider $H/Z(G) < G/Z(G)$

by induction get $\bar{N} < G/Z(G)$ with

$\bar{N}$ normalizing $H/Z(G)$.

consider N s.t. $N/Z(G) = \bar{N}$, then

N normalizes H since $\overline{nhn^{-1}} \in H/Z(G)$

$\bar{n}\bar{h}\bar{n}^{-1}$

but $H = \{g \in G \mid \bar{g} \in H/Z(G)\}$