S set, $R \subset W(S)$ some words

define $\langle S \mid R \rangle$ to be the group

$$F(S) \Big/ \begin{matrix}\text{smallest normal subgroup containing} \\ \text{the image of } R \text{ in } F(S)\end{matrix}$$

$$= F(S) / \langle R^G \rangle$$

<u>Notation:</u>

- if $T \subset G$ subset of a group $\langle T \rangle$ = smallest subgp containing T
- if $T, U \subset G$ subsets of a group

$$T^U = \{ u^{-1} t u \mid t \in T, u \in U \}$$

<u>note:</u> if $H < G$, $\langle H^G \rangle$ = smallest normal subgroup containing H

if $h_1, \dots, h_r \in H^G$ $\qquad h_i = g_i h_i' g_i^{-1}$

$$g h_1 \cdots h_r g^{-1} \overset{?}{\in} \langle H^G \rangle \qquad g_i \in G \;\; h_i' \in H$$

$$\parallel$$

$$(g h_1 g^{-1})(g h_2 g^{-1}) g \cdots g h_r g^{-1}$$

$$g h_i g^{-1} = g g_i h_i' g_i^{-1} g^{-1} \in H^G$$

<u>Remark</u>: if $T \subset G$ then $\langle T \rangle$ = image of $W(T)$ in G

$\langle \{\sigma, \tau\} \mid \{\tau\sigma\tau\sigma, \tau^2, \sigma^4\} \rangle$ omit $\{\ \}$

$= \langle \sigma, \tau \mid \tau\sigma\tau\sigma, \tau^2, \sigma^4 \rangle$

$= \langle \sigma, \tau \mid \tau\sigma\tau\sigma = 1, \tau^2 = 1, \sigma^4 = 1 \rangle$

$= \langle \sigma, \tau \mid \tau\sigma\tau^{-1} = \sigma^{-1}, \tau^2 = 1, \sigma^4 = 1 \rangle$

Our def $\neq$ book's def of free groups but they are canonically isomorphic via a universal property.

$\updownarrow$ object oriented programming (API)

CS

Let S be a set, $F(S) = W(S)/\sim$

$$S \to F(S)$$

Observation: if $S \xrightarrow{\varphi} G$ is a set map from S to a group

then $\exists!$ way to extend φ to a group hom.

$F(S) \longrightarrow G$.

Conversely a group hom. $F(S) \to G$ gives a set map $S \to F(S) \to G$

$$\begin{tikzcd} S \arrow[r] \arrow[rr, bend right] & F(S) \arrow[r] & G \end{tikzcd}$$

gives $\mathrm{Hom}_{sets}(S, G) \leftrightarrows \mathrm{Hom}_{gp}(F(S), G)$

in bijection.

This characterizes free group:

$F(S)$ is group w/ set map $S \to F(S)$ s.t.

$\forall$ set maps $S \to G$, $\exists!$ $F(S) \to G$

such that
$$\begin{tikzcd} S \arrow[rr] \arrow[dr] & & G \\ & F(S) \arrow[ur] & \end{tikzcd}$$
commutes

$$\begin{tikzcd} S \arrow[r] \arrow[d] & G \\ F(S) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

<u>Note:</u> If F is any other gp s.t. $S \to F$ set map, and $\forall\, S \to G$ set maps $\exists!$ $F \to G$ s.t.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & G \\ F \arrow[ur] & \end{tikzcd}$$

commutes $\Rightarrow$ $F \simeq F(S)$

univ prop for $F(S)$ w/ $G = F$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F \\ F(S) \arrow[ur, dashed, "\exists!\alpha"'] & \end{tikzcd}$$

univ prop F w/ $G = F(S)$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F(S) \\ F \arrow[ur, dashed, "\exists!\beta"'] & \end{tikzcd}$$

univ. prop for $F(S)$ w/ $G = F(S)$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F(S) \\ F(S) \arrow[ur, dashed, "\exists! = id"'] & \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & F \\ F \arrow[ur, "\exists! = id"'] & \end{tikzcd}$$

$$\begin{tikzcd} & F \\ S \arrow[ur] \arrow[r] \arrow[dr] & F(S) \arrow[u, "\alpha"] \\ & F \arrow[u, "\beta"] \arrow[uu, bend right, "= id"'] \end{tikzcd}$$

$\alpha\beta = id$

$$\begin{tikzcd} & F(S) \\ S \arrow[ur] \arrow[r] \arrow[dr] & F \arrow[u, "\beta"] \\ & F(S) \arrow[u, "\alpha"] \arrow[uu, bend right, "= id"'] \end{tikzcd}$$

$\beta\alpha = id.$

Given cats $\mathcal{C}, \mathcal{D}$, functors $F : \mathcal{C} \to \mathcal{D}$
$G : \mathcal{D} \to \mathcal{C}$

we say that F is left adjoint to G
or G is right adjoint to F or write
$F \dashv G$

if $\mathrm{Hom}_{\mathcal{D}}(FX, Y) = \mathrm{Hom}_{\mathcal{C}}(X, GY)$
we are given bijections which are natural in X & Y.

Given cat $\mathcal{C}$, can form "opposite category"

$\mathcal{C}^{op}$ same objects $\mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$

composition in opposite direction (s & t switched)

If $\mathcal{C}$ is a category, $X \in \mathrm{ob}(\mathcal{C})$
get a functor $\mathcal{C} \longrightarrow \mathrm{Sets}$
$Y \longmapsto \mathrm{Hom}(X, Y)$

<u>ex:</u> $\mathcal{C} = \mathrm{Gps}$ $X = \mathbb{Z}$ $\mathrm{Hom}(X, G) = G$ as a set.
$\mathrm{Hom}(\mathbb{Z}, -) =$ forgetfull functor.

$\mathcal{C}$ = gps $\quad X = \mathbb{Z} \times \mathbb{Z} \qquad \mathrm{Hom}(X, G)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad$ {pairs of commuting elmts}

Similarly get a functor

$$\mathcal{C}^{op} \longrightarrow \mathrm{Sets}$$

$$Y \longmapsto \mathrm{Hom}(Y, X)$$

if $\mathcal{C}$ & $\mathcal{D}$ are cats, can form new cat

$\mathcal{C} \times \mathcal{D} \qquad ob(\mathcal{C} \times \mathcal{D})$ = pairs (x, y) $\quad x \in ob(\mathcal{C})$, $y \in ob(\mathcal{D})$

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((x,y),(x',y')) = \mathrm{Hom}_{\mathcal{C}}(x,x') \times \mathrm{Hom}_{\mathcal{D}}(y,y')$$

<u>Note</u>: given cat $\mathcal{C}$, get a functor

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Set}$$

$$(X, Y) \longmapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

$$\left[(X,Y) \xrightarrow{(\alpha^{op},\beta)} (X',Y')\right] \longmapsto \left[\begin{array}{c}\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X',Y') \\ f \longmapsto \beta f \alpha\end{array}\right]$$

$\alpha : X' \to X$

$\beta : Y \to Y'$

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\alpha"'] \arrow[d, dashed] \\ Y \arrow[r, "\beta"'] & Y' \end{tikzcd}$$

Language: functor $\mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{E}$ "bifunctors"

Given functors $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$

get bifunctors

$$\begin{tikzcd} \mathcal{D}^{op} \times \mathcal{D} & \mathcal{C}^{op} \times \mathcal{D} \arrow[l, "F^{op}"'] \arrow[dl, "G"'] \arrow[r] & \mathrm{Sets} \\ \mathcal{C}^{op} \times \mathcal{C} & & \end{tikzcd}$$

$$X, Y \overset{\alpha}{\longmapsto} \mathrm{Hom}_{\mathcal{C}}(X, GY)$$

$$X, Y \overset{\beta}{\longmapsto} \mathrm{Hom}_{\mathcal{D}}(FX, Y)$$

We say $F \dashv G$ if $\exists\, \eta : \alpha \Rightarrow \beta$ a natural isom.

i.e. $\forall\, (X, Y) \xrightarrow{(f^{op}, g)} (X', Y')$

we have a comm. diagram

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, GY) \arrow[r, "\eta_{X,Y}"] \arrow[d, "\alpha(f^{op}, g)"] & \mathrm{Hom}_{\mathcal{D}}(FX, Y) \arrow[d, "\beta(f^{op}, g)"] \\
\mathrm{Hom}_{\mathcal{C}}(X', GY') \arrow[r, "\eta_{X',Y'}"'] & \mathrm{Hom}_{\mathcal{D}}(FX', Y')
\end{tikzcd}
$$