

$S$  set,  $R \subset W(S)$  some words

define  $\langle S | R \rangle$  to be the group

$F(S)$  / smallest normal subgroup containing  
the image of  $R$  in  $F(S)$   
 $= F(S) / \langle R^G \rangle$

Notation:

- if  $T \subset G$  subset of a group  $\langle T \rangle =$  smallest subgp containing  $T$
- if  $T, U \subset G$  subsets of a group

$$T^U = \{ u^{-1} t u \mid t \in T, u \in U \}$$

note: if  $H < G$ ,  $\langle H^G \rangle =$  smallest normal subgroup conby  $H$

if  $h_1, \dots, h_r \in H^G$

$$g h_1 \dots h_r g^{-1} \in \langle H^G \rangle$$

$$(g h_1 g^{-1} g h_2 g^{-1} g \dots g h_r g^{-1})$$

$$g h_i g^{-1} = g g_i h_i g_i^{-1} g^{-1} \in H^G$$

$$h_i = g_i h_i g_i^{-1}$$

$g_i \in G, h_i \in H$

Remark: if  $T \subset G$  then  $\langle T \rangle = \text{image of } W(T) \text{ in } G$

$$\langle \{\sigma, \tau\} \mid \{\tau\sigma\tau\sigma, \tau^2, \sigma^n\} \rangle \quad \text{omit } \{ \}$$

$$= \langle \sigma, \tau \mid \tau\sigma\tau\sigma, \tau^2, \sigma^n \rangle$$

$$= \langle \sigma, \tau \mid \tau\sigma\tau\sigma = 1, \tau^2 = 1, \sigma^n = 1 \rangle$$

$$= \langle \sigma, \tau \mid \tau\sigma\tau^{-1} = \sigma^{-1}, \tau^2 = 1, \sigma^n = 1 \rangle$$

Our def  $\neq$  book's def of free groups  
but they are canonically isomorphic via  
a universal property.

↕ object oriented programming (API)  
CS

Let  $S$  be a set,  $F(S) = W(S)/\sim$

$$S \rightarrow F(S)$$

Observation: if  $S \xrightarrow{\varphi} G$  is a set map from  $S$  to a group then  $\exists!$  way to extend  $\varphi$  to a group hom.

$$F(S) \rightarrow G.$$

Conversely a group hom.  $F(S) \rightarrow G$  gives

$$\text{a set map } S \rightarrow F(S) \rightarrow G$$

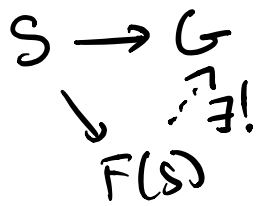
gives  $\text{Hom}_{\text{sets}}(S, G) \xleftrightarrow{\sim} \text{Hom}_{\text{gp}}(F(S), G)$   
in bijection.

This characterizes free groups:

$F(S)$  is group w/ set map  $S \rightarrow F(S)$  s.t.

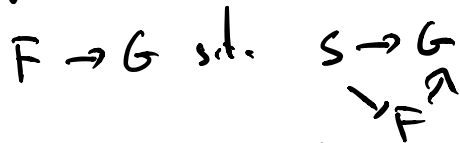
$\forall$  set maps  $S \rightarrow G$ ,  $\exists!$   $F(S) \rightarrow G$

such that 
$$\begin{array}{ccc} S & \xrightarrow{\quad} & G \\ \downarrow & & \uparrow \\ & F(S) & \end{array} \text{ commutes}$$



Note: If  $F$  is any other gp

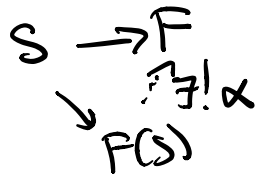
s.t.  $S \rightarrow F$  set map, and  
 $\nexists S \rightarrow G$  set maps  $\exists!$



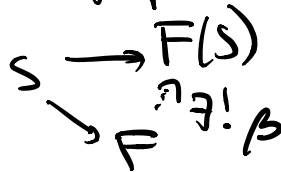
commutes  $\Rightarrow$

$$F \cong F(S)$$

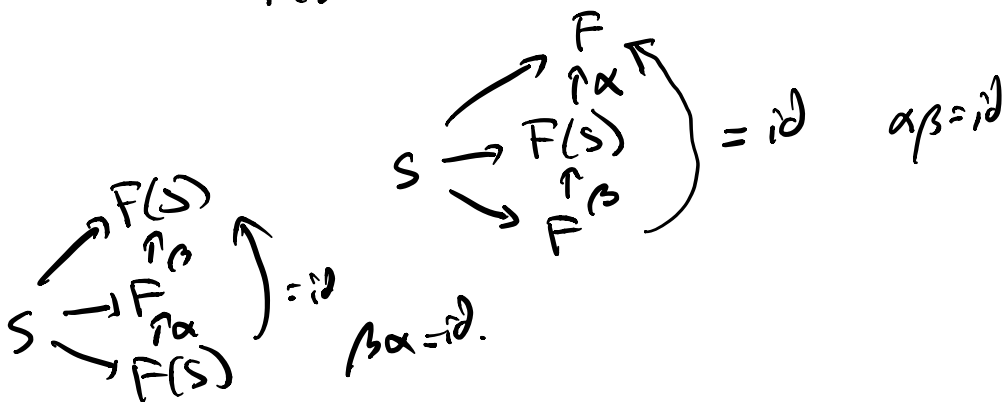
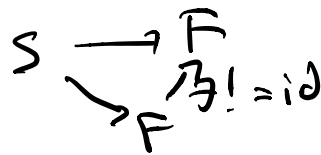
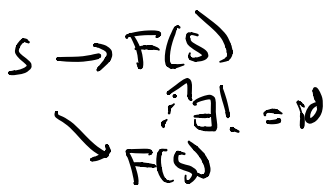
univ prop for  $F(S)$  w/  $G=F$



univ prop  $F$  w/  $G=F(S)$



univ. prop for  $F(S)$  w/  $G=F(S)$



Given cats  $\mathcal{C}, \mathcal{D}$ , functors  $F: \mathcal{C} \rightarrow \mathcal{D}$

$G: \mathcal{D} \rightarrow \mathcal{C}$

we say that  $F$  is left adjoint to  $G$

or  $G$  is right adjoint to  $F$  or write

$F \dashv G$

if  $\text{Hom}_{\mathcal{D}}(FX, Y) = \text{Hom}_{\mathcal{C}}(X, GY)$

we are given bijections which are natural in  $X$  &  $Y$ .

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Given cat  $\mathcal{C}$ , can form "opposite category"

$\mathcal{C}^{op}$  same objects  $\text{Hom}_{\mathcal{C}^{op}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$

composition in opposite direction (s.t. it switches)

If  $\mathcal{C}$  is a category,  $X \in \text{ob}(\mathcal{C})$

get a functor  $\mathcal{C} \rightarrow \text{Sets}$

$Y \mapsto \text{Hom}(X, Y)$

ex:  $\mathcal{C} = \text{ops}$   $X = \mathbb{Z}$

$\text{Hom}(X, G) = G$  as a set.

$\text{Hom}(\mathbb{Z}, -) =$  forgetful functor.

$$\mathcal{C} = \text{gps} \quad X = \mathbb{Z} \times \mathbb{Z} \quad \text{Hom}(X, G)$$

{pairs of commut  
diags}

Similarly get a functor

$$\mathcal{C}^{\text{op}} \longrightarrow \text{Set}$$

$$Y \longmapsto \text{Hom}(Y, X)$$

if  $\mathcal{C}$  &  $\mathcal{D}$  are cats, can form new cat

$$\mathcal{C} \times \mathcal{D} \quad \text{ob}(\mathcal{C} \times \mathcal{D}) = \text{pairs } (x, y) \quad \begin{array}{l} x \in \text{ob}(\mathcal{C}) \\ y \in \text{ob}(\mathcal{D}) \end{array}$$

$$\text{Hom}_{\mathcal{C} \times \mathcal{D}}((x, y), (x', y')) = \text{Hom}_{\mathcal{C}}(x, x') \times \text{Hom}_{\mathcal{D}}(y, y')$$

Note given cat  $\mathcal{C}$ , get a functor

$$\mathcal{C}^{\text{op}} \times \mathcal{C} \longrightarrow \text{Set}$$

$$(X, Y) \longmapsto \text{Hom}_{\mathcal{C}}(X, Y)$$

$$\left[ (x, y) \xrightarrow{(\alpha, \beta)} (x', y') \right] \longmapsto \left[ \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x', y') \right]$$

$\alpha: x' \rightarrow x$   
 $\beta: y \rightarrow y'$

$f \longmapsto \beta \circ f \circ \alpha$

$$\begin{array}{ccc}
 X & \xleftarrow{\alpha} & X' \\
 f \downarrow & & \downarrow \dots \\
 Y & \xrightarrow{\beta} & Y'
 \end{array}$$

Lay out: functors  $\mathcal{C} \times \mathcal{D} \xrightarrow{F} \mathcal{E}$  "bifunctors"

Green functors  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$

get bifunctors

$$\begin{array}{ccccc}
 \mathcal{D}^{\text{op}} \times \mathcal{D} & \xleftarrow{F^{\text{op}}} & \mathcal{C}^{\text{op}} \times \mathcal{D} & \longrightarrow & \text{Sets} \\
 & \searrow G & X, Y & \xrightarrow{\alpha} & \text{Hom}_{\mathcal{C}}(X, GY) \\
 \mathcal{C}^{\text{op}} \times \mathcal{C} & & & \searrow \beta & \text{Hom}_{\mathcal{D}}(FX, Y)
 \end{array}$$

We say  $F \dashv G$  if  $\exists \eta: \alpha \Rightarrow \beta$  a natural isom.

i.e.  $\forall (X, Y) \xrightarrow{(f^{\text{op}}, g)} (X', Y')$

we have a comm. diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, GY) & \xrightarrow{\eta_{X,Y}} & \text{Hom}_{\mathcal{D}}(FX, Y) \\ \downarrow \alpha(f^{\#}, g) & & \downarrow \beta(f^{\#}, g) \\ \text{Hom}_{\mathcal{C}}(X', GY') & \xrightarrow{\eta_{X',Y'}} & \text{Hom}_{\mathcal{D}}(FX', Y') \end{array}$$