Basic setup of sampling / quantization

Signal $f(t)$ $\qquad$ $g(x,y)$ etc

$C_{\mathbb{R}}([a,b])$ $\qquad$ $C_{\mathbb{R}}([a,b]\times[c,d])$

e.g. amplitude of signal $\qquad$ e.g. shading of parts of a picture

> **Def** $S \subset \mathbb{R}$ or $\mathbb{R}^2$ or $\mathbb{R}^n$ region
>
> $$C_{\mathbb{R}}(S) = \{ f : S \to \mathbb{R} \mid f \text{ is continuous} \}$$
>
> $$C_{\mathbb{C}}(S) = \{ \quad \cdot \qquad \mathbb{C} \mid \qquad \}$$

Note $C_{\mathbb{R}}(S) \subset C_{\mathbb{C}}(S)$ and it is often more convenient to think about complex #s.

We may write $C_F(S)$ where F can stand for $\mathbb{R}$ or $\mathbb{C}$ when we don't want to make a choice.

<u>Sampling</u>: choose sample points $t_0, t_1, \dots, t_{N-1} \in [a,b]$ or $(x_{ij} \in [a,b]\times[c,d])$,

and consider

$$\vec{y} = \begin{bmatrix} f(t_0) \\ f(t_1) \\ \vdots \\ f(t_{N-1}) \end{bmatrix}$$

a sampling of f

we think of $\vec{y}$ as an approximation of f

Quantization

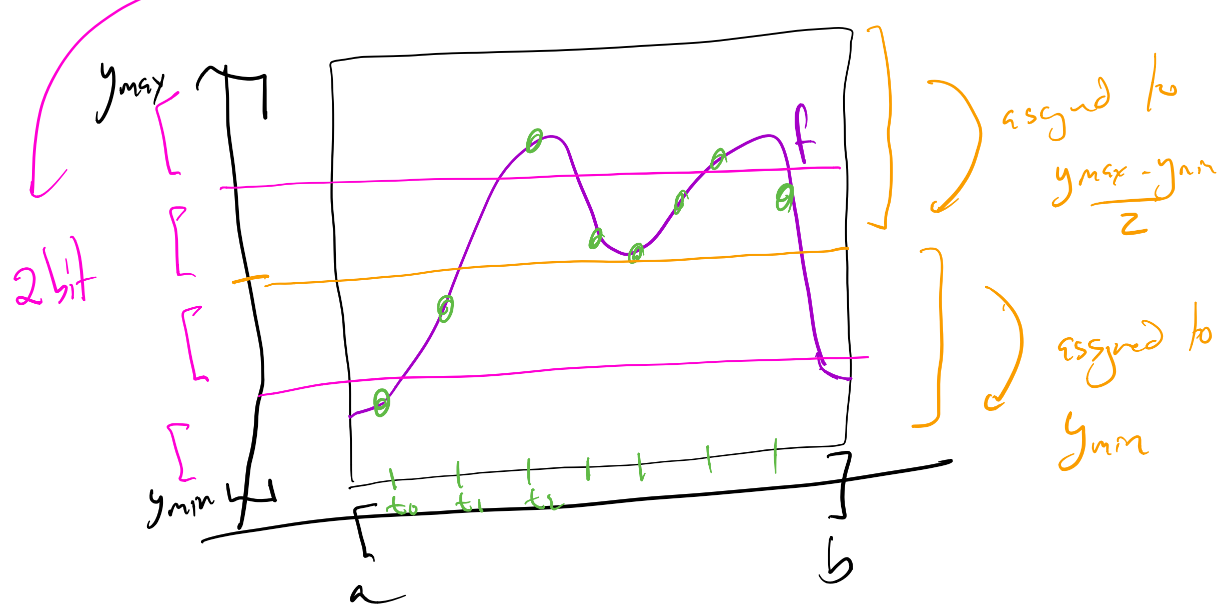
It's common to use binary quantization (binary digits)

So, if values of f range in some interval $[y_{min}, y_{max}]$, then

decided

1 bit quantization

2 bit quantization



if $y_{min} = 0$ $y_{max} = 1$

we assign a y value in an n-bit rep

to $\lfloor y \cdot 2^n \rfloor \cdot \frac{1}{2^n}$

In general:

$$\left\lfloor \underbrace{\left(\frac{y - y_{min}}{y_{max} - y_{min}} \right)}_{\substack{\text{between } 0 \,\&\, 1 \\ \text{encoding how high} \\ \text{in interval}}} 2^n \right\rfloor \cdot \underbrace{\left(\frac{y_{max} - y_{min}}{2^n} \right)}_{\text{rescale}} \underbrace{+ \, y_{min}}_{\text{shift back.}}$$

Vector spaces

Some useful vector spaces

$C_F(S)$ is an F-vector space

polynomials of degree $\le n$ is an $n+1$ dim'l vector space

Prop if f, g are polynomials of deg $\le n$ in x and have same values at distinct pts $x_0, \ldots, x_n$, then $f = g$

Pf: We can assume $F = \mathbb{C}$ since $=$ is $=$.

in this case, FTA $\Rightarrow$ $f - g = \prod_{i=1}^{n} (x - a_i)$ or 0

know that at x_i, $f(x_i) = g(x_i)$ $\Rightarrow$ $(f-g)(x_i) = 0$

$\Rightarrow$ $a_j = x_i$ some j get $n+1$ distinct factors

$\Rightarrow$ $f - g = 0$. $\square$.

Cor fix distinct pts $a_0, \dots, a_n \in F$

$$\{\text{polys of deg} \leq n\} \longrightarrow F^{n+1} \quad \text{via}$$

$$f \longmapsto \begin{bmatrix} f(a_0) \\ \vdots \\ f(a_n) \end{bmatrix}$$ is an isom of vectrspces.

just showed injective. for surjective,

if we have $b_0, \dots, b_n \in F^{n+1}$, define

$$f(x) = \sum_{i=0}^{n} \left[b_i \underbrace{\prod_{j \neq i} \left(\frac{x - a_j}{a_i - a_j} \right)}_{\substack{1 \text{ if } x = a_i \\ 0 \text{ if } x = a_k,\, k \neq i}} \right]$$

so surjective

not needed since f.d. vectr spcs same dim.

basis $1, x, \dots, x^n$

We recall, F^n has an inner product

$$(x_1, \dots, x_n) \cdot (y_1, \dots, y_n) \equiv \sum x_i y_i$$

alternately

$$= [x_1 \ \cdots \ x_n] \begin{bmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

Suppose we have a basis $\vec{b}_1, \dots, \vec{b}_n$ for F^n

i.e. $\vec{b}_i = \begin{bmatrix} b_{i1} \\ \vdots \\ \vdots \\ b_{in} \end{bmatrix}$

a <u>dual basis</u> is a basis of vects $\vec{f}_1, \dots, \vec{f}_n$ such that $\vec{f}_i \vec{b}_j = \delta_{ij}$

This is extremely useful in practice

Suppose we have a useful basis $\vec{b}_1, \dots, \vec{b}_n$ and some vector $\vec{v} \in F^n$

we know abstractly that we can write $\vec{v} = \sum \beta_i \vec{b}_i$ some β_i, but how do we find the β_i's?

if we know $\vec{v}$ & $\vec{f}_i$'s in standard basis for F^n, can compute $\vec{v} \cdot \vec{f}_j$

$$= \left(\sum \beta_i \vec{b}_i \right) \cdot \vec{f}_j = \sum \beta_i (\vec{b}_i \cdot \vec{f}_j)$$

$$= \beta_j !$$

So lets you rewrite in terms of $\vec{b}_i$'s.

matrix notation

$$\vec{b}_i \cdot \vec{f}_j = \delta_{ij} \text{ means}$$

write $\vec{f}_j = (f_{1j}, \dots, f_{nj})$ as a column

$$\begin{bmatrix} f_{1j} & \cdots & f_{nj} \end{bmatrix} \begin{bmatrix} b_{i1} \\ \vdots \\ b_{in} \end{bmatrix} = \delta_{ij}$$

but this is the matrix equation

$$\begin{bmatrix} f_{11} & f_{21} & \cdots & f_{n1} \\ f_{12} & f_{22} & \cdots & f_{n2} \\ \vdots & & & \vdots \\ f_{1n} & \cdots & \cdots & f_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{21} & \cdots & b_{n1} \\ b_{12} & b_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix} = I_n$$

lets read this in linr algebra

given a basis $\vec{b}_1, \ldots, \vec{b}_n$, B = matrix with

$\vec{b}_i$'s as columns. this is the change of basis

matrix : $\{\vec{e}_i\} \xrightarrow{B} \{\vec{b}_i\}$