Wavelets

A particular choice or method for constructing bases of our V.space of signals

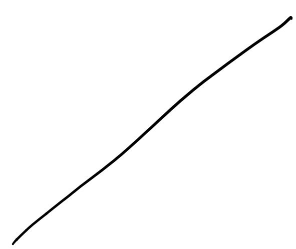
two collections of basis vectors

expected	surprising
trend	detail
average	error

Can use these to separate out "noise"

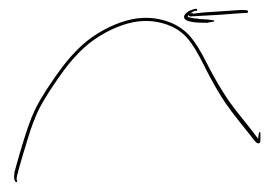
use to detect "features"

Example: "Haar wavelet transform"



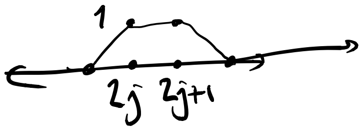
← everything a surprise

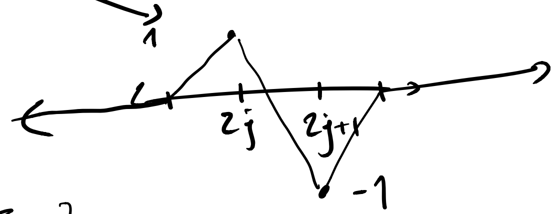
← no surprises



Actual transform:

basis elements





$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Wavelet transform procedure:

"average"

$$S \xrightarrow[\text{break into even/odd}]{} \begin{bmatrix} S_{even} \\ S_{odd} \end{bmatrix} \xrightarrow[\text{transform}]{} \begin{bmatrix} S_{prediction} \\ S_{detail} \end{bmatrix}$$

error

① gather information we need to combine w/ odd part to get prediction. put in 1

$$\begin{bmatrix} S_{useful\ info} \\ S_{odd} \end{bmatrix}$$

② compute error we would have predicting S_{odd} from this. put answer in 2

$$\begin{bmatrix} S_{useful} \\ S_{error} \end{bmatrix}$$

③ update 1 w/ prediction

$$\begin{bmatrix} S_{prediction} \\ S_{error} \end{bmatrix}$$

④ rescale

ignore scaling factors.

$\boxed{\text{Haar}}$

$$S \rightsquigarrow \begin{bmatrix} S_{even} \\ S_{odd} \end{bmatrix} \xrightarrow[\text{1}]{\text{nothing}} \begin{bmatrix} S_{even} \\ S_{odd} \end{bmatrix} \xrightarrow[\text{subtract first from second}]{\text{2 prediction}} \begin{bmatrix} S_{even} \\ S_{odd} - S_{even} \end{bmatrix}$$

$$= \begin{bmatrix} S_{even} \\ -2\, S_{error} \end{bmatrix}$$

3 update: add half the second to first

$$\begin{bmatrix} S_{avg} \\ -2\, S_{error} \end{bmatrix}$$

4 normalization: mult second by $-\frac{1}{2}$

$$\begin{bmatrix} S_{avg} \\ S_{error} \end{bmatrix}$$

$$S_{avg}[k] = \frac{S_{even}[k] + S_{odd}[k]}{2}$$

prediction

$$= \frac{S[2k] + S[2k+1]}{2}$$

$$-S_{error}[k] = S_{odd}[k] - S_{avg}[k]$$

$$= \frac{S_{odd}[k] - S_{even}[k]}{2}$$

$$S_{error}[k] = \frac{S_{even}[k] - S_{odd}[k]}{2}$$

Example

$$S = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \\ -2 \\ 3 \\ 3 \\ 4 \end{bmatrix} \rightarrow S_{even} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \end{bmatrix} \quad S_{odd} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{pmatrix} S_{even} \\ S_{odd} \end{pmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \\ \hline 2 \\ 5 \\ 3 \\ 4 \end{bmatrix} \xrightarrow{2} \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \\ \hline 2-0 \\ 5-1 \\ 3-(-2) \\ 4-3 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \\ \hline 2 \\ 4 \\ 5 \\ 1 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow \text{even} \\ \\ \\ \\ \leftarrow \text{2error} \end{array}$$

$$\xrightarrow{3} \begin{bmatrix} 0+1 \\ 1+2 \\ -2+2.5 \\ 3+0.5 \\ \hline 2 \\ 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 3.5 \\ \hline 2 \\ 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 3.5 \\ \hline -1 \\ -2 \\ -2.5 \\ -0.5 \end{bmatrix}$$

gives new expression of signal

$$1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + 0.5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 3.5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$-1\begin{bmatrix}1\\-1\\0\\\vdots\\0\end{bmatrix}-2\begin{bmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{bmatrix}-2.5\begin{bmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{bmatrix}-0.5\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\-1\end{bmatrix}$$

in "Haar wavelet basis"

$\mathbb{R}^2$ $\begin{bmatrix}x_0\\x_1\end{bmatrix}$ $\quad e_1 \quad e_2 \quad\quad e_1+e_2 \quad e_1-e_2$

$$\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\qquad\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}$$

$$ae_1+be_2=\left(\frac{a+b}{2}\right)(e_1+e_2)+\left(\frac{a-b}{2}\right)(e_1-e_2)$$

$$\begin{bmatrix}\frac{a+b}{2}\\\frac{a+b}{2}\end{bmatrix}+\begin{bmatrix}\frac{a-b}{2}\\-\left(\frac{a-b}{2}\right)\end{bmatrix}=\begin{bmatrix}2a/2\\2b/2\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

coeff of e_1+e_2 = avg of coeffs

coeff of e_1-e_2 = $\dfrac{(\text{coeff of } e_1) - (\text{coeff of } e_2)}{2}$

<u>Haar transform:</u>

$$\begin{bmatrix} s[0] \\ \vdots \\ s[2N-1] \end{bmatrix} \rightsquigarrow \begin{bmatrix} s_{avg} \\ \hline s_{error} \end{bmatrix} = \begin{bmatrix} \frac{s[0]+s[1]}{2} \\ \frac{s[2]+s[3]}{2} \\ \vdots \\ \hline \frac{s[0]-s[1]}{2} \\ \frac{s[2]-s[3]}{2} \\ \vdots \end{bmatrix}$$

$$\begin{bmatrix} s[0] \\ s[1] \end{bmatrix} = \left(\frac{s[0]+s[1]}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{s[0]-s[1]}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The reason for interpreting output of Haar transform as coeffs in new basis.

$$S = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \\ -2 \\ 3 \\ 3 \\ 4 \end{bmatrix} \xrightarrow{\text{Haar}} \begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 3.5 \\ \hline -1 \\ -2 \\ -2.5 \\ -0.5 \end{bmatrix}$$

repeat process:

$$\begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 3.5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0.5 \\ \hline 3 \\ 3.5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} (1+3)/2 \\ (0.5+3.5)/2 \\ \hline (1-3)/2 \\ (0.5-3.5)/2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ \hline -1 \\ -1.5 \end{bmatrix}$$

$$\begin{bmatrix}1\\3\\0.5\\3.5\end{bmatrix} = 2\begin{bmatrix}1\\1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\\1\end{bmatrix} - 1\begin{bmatrix}1\\-1\\0\\0\end{bmatrix} - 1.5\begin{bmatrix}0\\0\\1\\-1\end{bmatrix}$$

"plug into first expression"

$$1\begin{bmatrix}1\\1\\0\\0\\0\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\\1\\0\\ \vdots\end{bmatrix} + 0.5\begin{bmatrix}\ \end{bmatrix} + 3.5\begin{bmatrix}\ \end{bmatrix}$$

$$= 2\begin{bmatrix}1\\1\\1\\1\\0\\0\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\0\\0\\1\\1\\1\\1\end{bmatrix} - 1\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix} - 1.5\begin{bmatrix}0\\0\\0\\0\\1\\1\\-1\\-1\end{bmatrix}$$