Quick Haar Review

Notation: s_n signal w/ $N = 2^n$ sample pts $(M = 2^{n-1})$

(1-Step) Haar transform:

$$s_n \longmapsto s_{n-1}, d_{n-1}$$

s_n: length N

s_{n-1}, d_{n-1}: length M

s_{n-1} "trend"

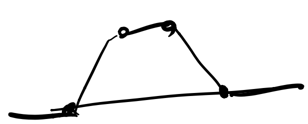
d_{n-1} "detail"

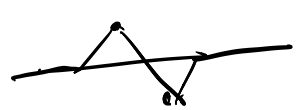
$$s_{n-1}[k] = \frac{1}{2}\left(s_n^{even}[k] + s_n^{odd}[k]\right) = \frac{1}{2}\left(s_n[2k] + s_n[2k+1]\right)$$

$$d_{n-1}[k] = s_{n-1}[k] - s_n[2k+1] = s_n[2k] - s_{n-1}[k]$$
$$= \frac{1}{2}\left(s_n[2k] - s_n[2k+1]\right)$$

Corresponds to coeffs in new "Haar basis"

$$h_k = e_{2k} + e_{2k+1} \qquad h_k' = e_{2k} - e_{2k+1}$$





Could continue transform $s_{n-1} \rightarrow s_{n-1}, d_{n-2}$

Today: Introduce some new wavelets

Wavelet signal — trend (expected)

— detail (surprising/interesting)

Haar: any change is surprising (expect constant)

More interesting: Weighted average (Daub 4)

Averaging at diff. pts (CDF(2,2))

use a weighted avg of seq of pts before.

val at odd ~ average at surrounding pts.

CDF(2,2)

$x \sim x[0], \dots x[N-1]$ $N = 2M$

x_{even} x_{odd}

Idea: predict $x[2k+1] \sim \dfrac{x[2k] + x[2k+2]}{2}$

To make this make sense, we'll assume x is periodic

Procedure:

$$\underset{\text{split}}{x \rightsquigarrow} \begin{bmatrix} x_{even} \\ x_{odd} \end{bmatrix} \xrightarrow[P]{\text{predict}} \begin{bmatrix} x_{even} \\ d \end{bmatrix} \xrightarrow[U]{\text{update}} \begin{bmatrix} s \\ d \end{bmatrix} \xrightarrow[D]{\text{normalize}}$$

normalize
mult top by $\sqrt{2}$
bott. $\frac{\sqrt{2}}{2}$

$$d = x_{odd} - \frac{1}{2}\left(x_{even} + S^{-1} x_{even}\right)$$

$$d[k] = x[2k+1] - \frac{1}{2}\left(x_{even}[k] + S^{-1} x_{even}[k]\right)$$

$$x[2k+1] - \frac{1}{2}\left(x_{even}[k] + x_{even}[k+1]\right)$$

$$x[2k+1] - \frac{1}{2}\left(x[2k] + x[2k+2]\right)$$

$$s[k] = x_{even}[k] + \frac{1}{4}\left(d[k] + d[k-1]\right)$$

$$= x[2k] + \frac{1}{4}\left(x[2k+1] - \frac{1}{2}x[2k] - \frac{1}{2}x[2k+2]\right)$$

$$+ \frac{1}{4}\left(x[2k-1] - \frac{1}{2}x(2k-2) - \frac{1}{2}x[2k]\right)$$

$$= \frac{3}{4}x[2k] + \frac{1}{4}x[2k-1] + \frac{1}{4}x[2k+1] - \frac{1}{8}x[2k-2] - \frac{1}{8}x[2k+2]$$

<u>Roughly</u>: surprise is when value at $2k$ is not well reflected by adjacent values, but (less so) is well reflected at a further distance.

We've described a matrix (implicitly)

$$x \rightsquigarrow \begin{bmatrix} s \\ d \end{bmatrix} \qquad N \times N \text{ matrix } T_a$$

"analysis"

inverse matrix T_s "sythesis"

<u>Remark</u>: if T is any linear transformation can think of it as a change of basis from standard e_i to "new basis" b_i.

How to get b_i's?

$$T = \left[Te_0 \,\middle|\, Te_1 \,\middle|\, Te_2 \,\middle|\, \cdots \,\middle|\, Te_{N-1} \right]$$

$$Tv = \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_{N-1} \end{bmatrix} \text{ "means" } \qquad v = \begin{bmatrix} \vdots \end{bmatrix}$$

$$v = \sum_i \lambda_i b_i$$

T^{-1} = chage of basis from b_i's to e_i's

and so as above columns of T^{-1} are b_i's written in terms of the basis e_i's.

If T is any invertible matrix, which we want to think of as a chage of basis from $e_i \rightsquigarrow b_i$ then we can express b_i's in terms of our original basis as the columns of T^{-1}.

For CDF(2,2) we have

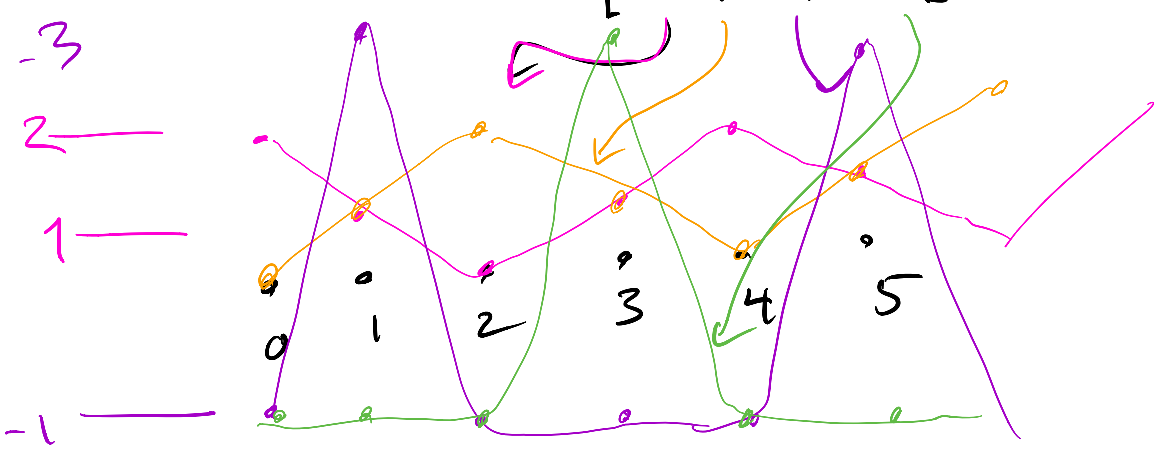
$$T_a = D\,U\,P\,\boxed{\text{split}}$$

if $N=4$:

$$T_a = \frac{1}{2\sqrt{2}}\begin{bmatrix} 3 & 1 & -1 & 1 \\ -1 & 1 & 3 & 1 \\ -1 & 2 & -1 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

$$T_s = \frac{\sqrt{2}}{4}\begin{bmatrix} 2 & 0 & -1 & -1 \\ 1 & 1 & 3 & -1 \\ 0 & 2 & -1 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

"1-step"
"1-scale"



We can repeat this!

$$x \longrightarrow \begin{bmatrix} s \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} s' \\ d' \end{bmatrix}$$, multiscale transforms

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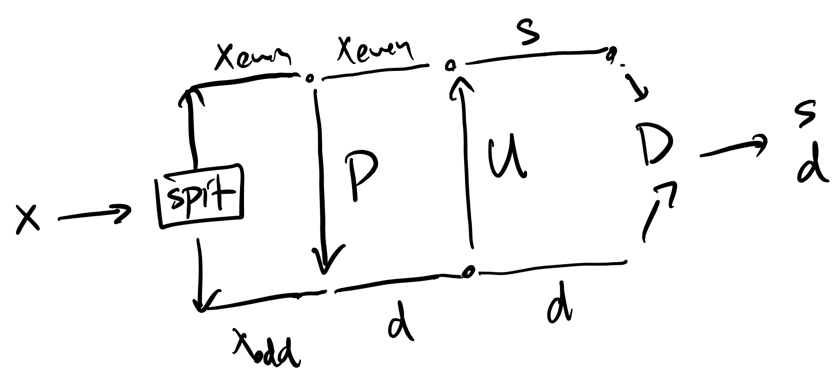
CDF(2,2)



Daub4