Applied Algebra Practice Sheet for Exam 2

This sheet is not meant to be exhaustive, but rather as a supplement to the problems from the homework since the last exam.

1.

Suppose T_a is the analysis matrix in the case $N = 4$, for a wavelet transform with scaling vector $(1, 2, 0, 0)$ and wavelet vector of $(2, -3, 1, 0)$. Compute $T_a(1, 1, 1, 1)$ and $T_a(1, -1, 0, 0)$.

2.

Suppose we have a wavelet transform given by $x \mapsto (s, d)$ where

$$d[k] = x[2k+1] - x[2k] - 2x[2k+2]$$

and

$$s[k] = x[2k] + d[k] + 3d[k-1].$$

Find block matrices P, U such that $T_a = UP\boxed{\text{split}}$.

3.

For the wavelet transform in the previous problem, find the scaling and wavelet vectors in the case $N = 4$.

4.

Recall the two-scale Haar Transform $x \mapsto (s_1, s_2, d_2)$. This is given by first performing the Haar transform $x \mapsto (s_1, d_1)$, then performing a second Haar transform $s_1 \mapsto (s_2, d_2)$ on the first trend, and collecting this all in the vector (s_1, s_2, d_2). Give a description, in block form, for the matrix which gives this linear transformation $x \mapsto (s_1, s_2, d_2)$.

Hint: you may need more than 4 blocks!

5.

Consider the following procedures, where T_a, T_s are the analysis and synthesis matrices for the Haar transform.

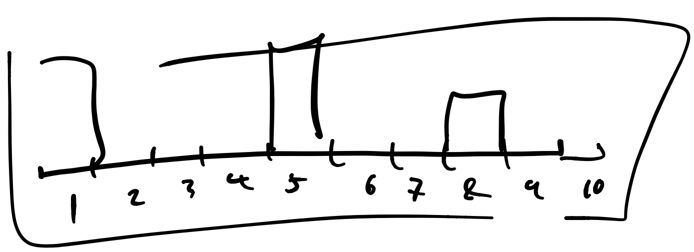
1 - apply T_a, look at the coordinates of the resulting vector and set to 0 all coordinates which are sufficiently small. Then apply T_s to the resulting vector
2 - apply T_a to get s, d, replace d with 0, then apply T_s.
3 - apply T_a to get s, d, replace s with 0, then apply T_s.

Consider the following goals we might have:

chnge coeff in Haar
prop to size of
$\int |f - f_{new}|^2 dx$

a - remove noise from a signal
b - find jumps in signal
c - compress signal

Which of the above operations would be potentially useful for these goals?



val 1 → 1
val 5 → 5
val 8 → 8

10 spaces

$x_1 = \ldots$
$x_2 = \ldots$

$$x \xmapsto{\text{Haar}} \begin{bmatrix} s_1 \\ d_1 \end{bmatrix} \begin{matrix} \leftarrow 2^{\ell-1} \end{matrix} \xrightarrow{\text{Haar}} \begin{bmatrix} s_2 \\ d_2 \end{bmatrix} \begin{matrix} \leftarrow 2^{\ell-2} \end{matrix} \longrightarrow \begin{bmatrix} s_2 \\ d_2 \\ d_1 \end{bmatrix} \begin{matrix} \leftarrow 2^{\ell-2} \\ \leftarrow 2^{\ell-1} \end{matrix}$$

2^ℓ

$$\begin{array}{r} 2^{\ell-2} \left\{ \begin{matrix} \\ \\ \end{matrix} \right. \\ 2^{\ell-1} \left\{ \begin{matrix} \\ \\ \\ \end{matrix} \right. \end{array} \begin{bmatrix} \text{———} \\ \text{———} \\ \\ \\ \end{bmatrix} x = \begin{bmatrix} s_2 \\ \hline d_2 \\ \hline d_1 \end{bmatrix}$$

$$x \longmapsto \begin{bmatrix} s \\ d \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \text{———} \\ \text{- - -} \\ \text{———} \end{bmatrix}}_{T_a} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{2m-1} \end{bmatrix} = \begin{bmatrix} \boxed{s_0} \\ \vdots \\ s_{m-1} \\ \hline \boxed{d_0} \\ \vdots \\ d_{m-1} \end{bmatrix}$$

Step 1: Remember How to write Haar Wavelet transform.

$$T_a = \left[\begin{array}{c}\mathcal{U}\\ \hline \mathcal{V}\end{array}\right] = \begin{bmatrix}\frac{1}{2}I & \frac{1}{2}I\\ \frac{1}{2}I & -\frac{1}{2}I\end{bmatrix} \boxed{\text{split}}$$

$$T_a x = \begin{bmatrix}s\\ d\end{bmatrix} \Rightarrow \begin{array}{l} s = \mathcal{U}x \\ d = \mathcal{V}x \end{array}$$

$$x \longrightarrow \begin{bmatrix}s_2\\ d_2\\ d_1\end{bmatrix} \longleftarrow \text{"original } d\text{"} = \mathcal{V}x$$

$\Rightarrow$ matrix that does this

also has a $\mathcal{V}$ in the bottom half.

2^{ℓ} Haar

$$\left[\begin{array}{c}\mathcal{U}\\ \hline \mathcal{V}\end{array}\right]$$

$$\underset{2^{\ell}}{x} \xrightarrow[\text{Haar}]{} \begin{bmatrix}s_1\\ d_1\end{bmatrix} \leftarrow 2^{\ell-1}$$

$s_1 = \mathcal{U}x$

$$\left[\begin{array}{c}\mathcal{U}'\\ \hline \mathcal{V}'\end{array}\right] \; 2^{\ell-1} \text{ Haar}$$

$$\xrightarrow{\text{Haar}} \begin{bmatrix}s_2\\ d_2\end{bmatrix} \leftarrow 2^{\ell-2}$$

$$\Longrightarrow \begin{bmatrix}s_2\\ d_2\\ d_1\end{bmatrix} \begin{array}{l}\leftarrow 2^{\ell-2}\\ \\ \leftarrow 2^{\ell-1}\end{array}$$

$$x \longrightarrow \left[\begin{array}{c}\mathcal{U}x\\ \hline \mathcal{V}x\end{array}\right] \longrightarrow \left[\begin{array}{c}\mathcal{U}'\mathcal{U}x\\ \hline \mathcal{V}'\mathcal{U}x\\ \hline \mathcal{V}x\end{array}\right] = \left[\begin{array}{c}s_2\\ \hline d_2\\ \hline d_1\end{array}\right]$$

$$
\begin{bmatrix} \\ \\ \\ \end{bmatrix} \qquad T_a \begin{bmatrix} \\ \\ \\ \end{bmatrix} T_a^t
$$

x $\qquad\qquad$ $x[j,k]$

$$
T_a \begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \text{trend} \\ \hline \text{detail} \end{bmatrix}
\qquad
\left[\begin{array}{c|c}
\text{vert. trend/} & \text{vert trend} \\
\text{hor. trend} & \text{horiz detail} \\
\hline
\text{hor. trend} & \text{hor detail} \\
\text{vert detail} & \text{vert detail}
\end{array}\right]
$$