Today: • Wavelet transforms for 2-D signals

• Preview of wavelets from filter banks

$s[k] \quad k = 0, \ldots, N-1$

$$\vec{s} = \begin{bmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{bmatrix}$$

$\mathbb{R}^N$ or $\mathbb{C}^N$

2-dim'l signal

$s[j,k] \quad j,k = 0, \ldots, N-1$

$$s = \begin{bmatrix} s[0,0] & s[0,1] & \cdots & s[0,N-1] \\ \vdots & & & \\ s[N-1,0] & - & - & s[N-1,N-1] \end{bmatrix}$$

$\mathbb{R}^{N^2}$ or $\mathbb{C}^{N^2}$

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know: when we write vectors as columns $[\,]$ $\mathbb{C}^N$

lin transformations can write as

matrices $\mathbb{C}^{N^2}$

when we write vectors as matrices $\begin{bmatrix} & & \end{bmatrix}$ $\mathbb{C}^{N^2}$

space of lin transformations

is now $\mathbb{C}^{N^4}$

awkward to write down:

But certain nice kinds of lin transformations

are easier to describe.

$$\mathbb{C}^{N^2} \longleftrightarrow M_N(\mathbb{C})$$

given $A, B \in M_N(\mathbb{C})$ get a lin transformation

$$M_N(\mathbb{C}) \xrightarrow{f} M_N(\mathbb{C}) \text{ via}$$

$$T \longmapsto ATB = f(T)$$

$$T + T' \longmapsto \underbrace{A(T+T')B}_{f(T+T')} = \underbrace{ATB + AT'B}_{f(T)+f(T')}$$

$$\lambda T \longmapsto \underbrace{A(\lambda T)B}_{f(\lambda T)} = \lambda ATB = \lambda f(T)$$

2-d wavelet transforms are given by these kinds of maps:

given a matrix $S = \begin{bmatrix} & & \end{bmatrix}_{N \times N}$

thinking of this as a signal,

can consider each column as a 1-d signal

$$\begin{bmatrix} s[*,0] \,\big|\, s[*,1] \,\big|\, \cdots \end{bmatrix}$$

can apply T_a to each column.

$$\underbrace{\begin{bmatrix} s[*,0] \,\big|\, s[*,1] \,\big|\, \cdots \end{bmatrix}}_{S} \rightsquigarrow \underbrace{\begin{bmatrix} T_a s[*,0] \,\big|\, T_a s[*,1] \,\big|\, \cdots \end{bmatrix}}_{T_a S}$$

$$T \begin{bmatrix} v_0 \,\big|\, v_1 \,\big|\, v_2 \cdots \big|\, v_{N-1} \end{bmatrix} = \begin{bmatrix} T v_0 \,\big|\, T v_1 \,\big|\, \cdots \,\big|\, T v_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} w_0 \\ \hline w_1 \\ \hline \vdots \\ \hline w_{N-1} \end{bmatrix} T^t = \begin{bmatrix} Tw_0 \\ \hline Tw_1 \\ \hline \vdots \\ \hline Tw_{N-1} \end{bmatrix}$$

$$\begin{bmatrix} s[0,*] \\ \hline s[1,*] \\ \hline \vdots \\ \hline s[N-1,*] \end{bmatrix} T_a^t = \begin{bmatrix} T_a s[0,*] \\ \hline T_a s[1,*] \\ \hline \vdots \\ \hline T_a s[N-1,*] \end{bmatrix}$$

<u>Def</u>: the 2-d wavelet transform of s is

$$T_a s T_a^t = \left[\begin{array}{c|c|c|c} t_0^v & t_1^v & \cdots & t_{N-1}^v \\ \hline d_0^v & d_1^v & \cdots & d_{N-1}^v \end{array}\right] T_a^t$$

$$= \left[\begin{array}{c|c} \text{hor \& vert trend} & \text{horiz detail of vert trend} \\ \hline \text{vertical detail of horiz trend} & \text{vert detail of vertical detail} \end{array}\right]$$

$$\approx \begin{bmatrix} \text{blurred or denoised image} & \text{vertical features} \\ \text{horiz features} & \text{diagonal features} \end{bmatrix}$$

Step 6 : shift focus from periodic & finite fixed length signals

to signals of arbitrary length.

Punchline: all tools we've developed transfer painlessly to this context.

Nice language (transferred from DFT) to talk about frequencies arising in signals of arbitrary length

"Frequency spectrum"?

Filters in this context. (still described by (linear) convolution)

Step 2: Wavelet transforms for these.

$$x \xmapsto{T_a} \begin{bmatrix} s \\ d \end{bmatrix}$$

filtered signal (filter h_0)

$$\begin{tikzcd} & h_0 x \arrow[r] & \boxed{2\downarrow} h_0 x = s \\ x \arrow[ur] \arrow[dr] & & \\ & h_1 x \arrow[r] & \boxed{2\downarrow} h_1 x = d \end{tikzcd}$$

another filter

downsampling (forget odd)

The search for wavelets with good (desirable properties

$\Longleftrightarrow$ the search for good pairs of filters (h_0, h_1)