Today: towards (perhaps through 1.8)

Exponential function

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n + \dots$$

magical properties:

$$e^x e^y = \left(1 + x + \frac{1}{2}x^2 + \dots\right)\left(1 + y + \frac{1}{2}y^2 + \dots\right)$$

$$= \left(1 + (x+y) + \left(\frac{1}{2}x^2 + xy + \frac{1}{2}y^2\right) + \dots\right)$$

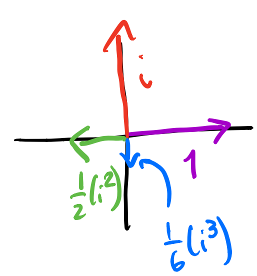
$$= \left(1 + (x+y) + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 + \dots\right)$$

$$= e^{x+y}$$

$$(e^x)^y = e^{xy}$$

can define e^i $\quad e^{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}$

Example e^i

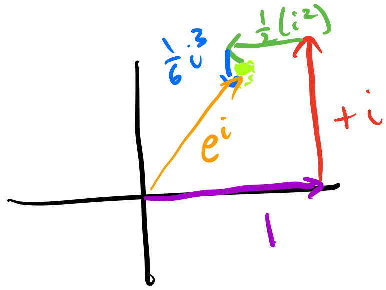


1	i	i^2	i^3	i^4
1	i	-1	$-i$	1

$$1 + i + \frac{1}{2}(i)^2 + \frac{1}{6}(i^3) + \frac{1}{24}(i^4)$$

e^{i}

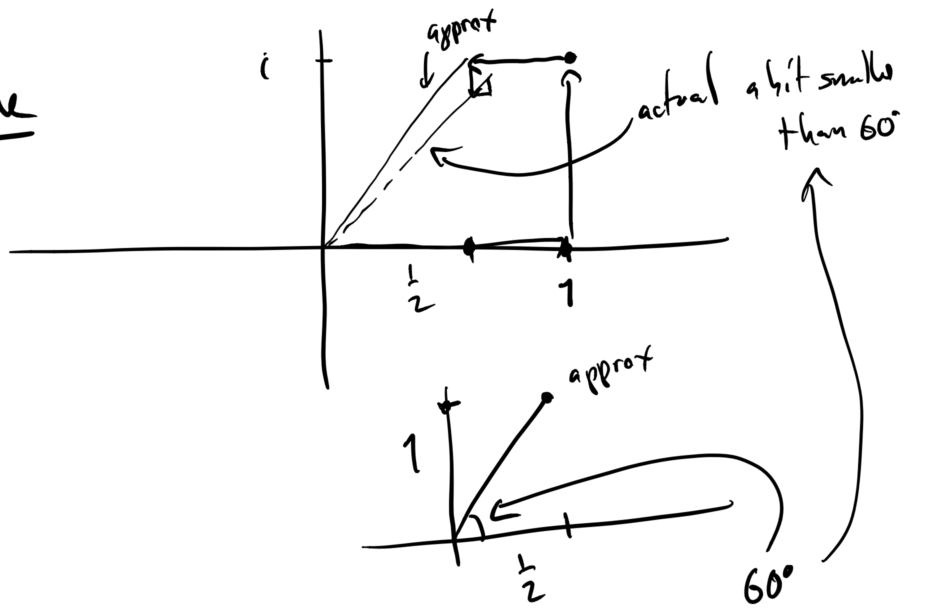
magnitude & direction?



$\|z\| = z\bar{z}$ $\qquad e^{i}\,\overline{e^{i}} = e^{i}e^{-i} = e^{0} = 1$

$$\overline{e^{z}} = \overline{\left(1 + z + \tfrac{1}{2}z^2 + \tfrac{1}{6}z^3 + \dots\right)}$$
$$= \bar{1} + \bar{z} + \tfrac{1}{2}\bar{z}^2 + \dots\Big) = e^{\bar{z}}$$

<u>angle</u>



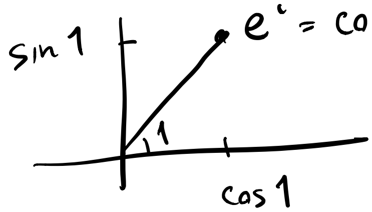
$\frac{\pi}{3} \approx \frac{3.14\ldots}{3}$

e^{i} length 1, angle a bit less than $\frac{3.14\ldots}{3}$

Magical answer: e^i has angle of $\underline{1 \text{ radian}}$

$$e^i =$$

$e^i = \cos 1 + i \sin 1$



$zw \leftarrow$ length = length z · length w

$\measuredangle = \measuredangle z + \measuredangle w$

z^a ← real #

length $= (\text{length } z)^a$

$\measuredangle = a \cdot (\measuredangle \text{ of } z)$

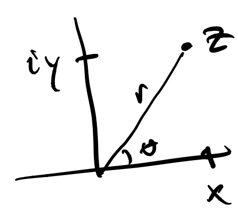
θ a real #

$$(e^i)^\theta = e^{i\theta} = \text{length } 1$$

$$\text{angle} = (\underbrace{\text{angle of } e^i}_{1})\,\theta = \theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$ $\underline{\text{Euler's formula}}$

given complex number $\quad z = x + iy$



$$\left.\begin{aligned} z &= re^{i\theta} \\ &= r(\cos\theta + i\sin\theta) \\ &= r\cos\theta + ir\sin\theta = x + iy \end{aligned}\right\} \quad \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$(r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

<u>Practice</u> $\cos 2\theta$ = real part of $e^{2i\theta}$ $= \cos 2\theta + i\sin 2\theta$

$$e^{2i\theta} = (e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

$$\cos 2\theta = Re\{e^{2i\theta}\} = Re\{\underbrace{\cos^2\theta - \sin^2\theta}_{} + 2i\cos\theta\sin\theta\}$$

$$= \cos^2\theta - \sin^2\theta$$

<u>Notation</u>: if $z = x+iy$, $Re\{z\} = x$ $Im\{z\} = y$

$$\int e^x \cos 2x \, dx = \int Re\{e^x e^{2ix}\} dx$$

$$= \int Re\{e^{(2i+1)x}\} dx$$

$$\cos 2x = Re\{e^{2ix}\}$$

$$= Re\left\{\int e^{(2i+1)x} dx\right\}$$

$$= Re\left\{\frac{1}{1+2i} e^{(2i+1)x}\right\}$$

$$Re\left\{\left(\frac{1-2i}{1-2i}\right)\frac{1}{1+2i} e^x e^{2ix}\right\} = Re\left\{\frac{1-2i}{1+4} e^x(\cos 2x + i\sin 2x)\right\}$$

$$= \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x + C.$$

Relevant Fact

$$\int \big(f(x) + i g(x)\big)\,dx = \int f(x)\,dx + i \int g(x)\,dx$$

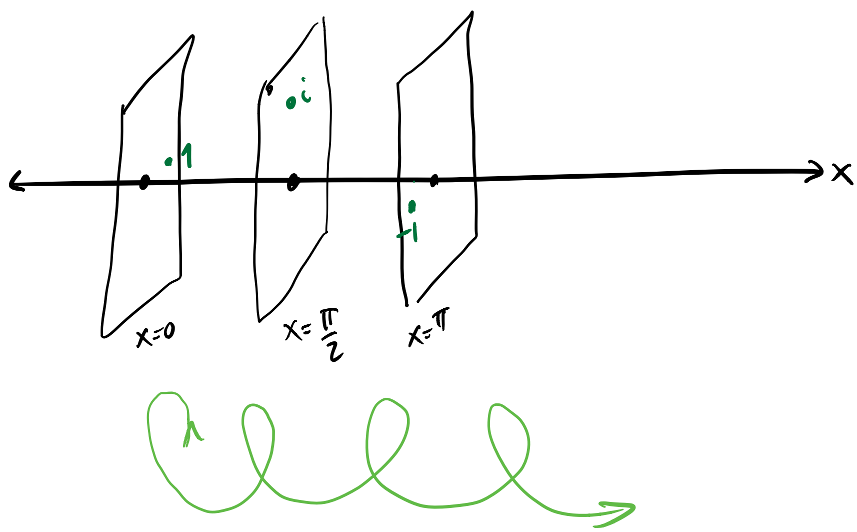
answer

What does $\int h(x)\,dx$ mean?

h complex valued $h = f + ig$

$\int \mathrm{Re} = \mathrm{Re}\{\int\}$

Graph of $f(x) = e^{ix}$



Convenient formulas

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Fourier Transform

start with a periodic function $f(x)$ $\left(f(x) = f(x+1) \right)$

would like to express $f(x)$ as a sum of sines & cosines

$\sin(2\pi x)$ $\quad \cos(2\pi x) \longleftrightarrow e^{2\pi i x}$

$\vdots$

$\sin(4\pi x)$ $\quad \cos(2\pi n x) \longrightarrow e^{2\pi i n x}$

$\sin(6\pi x)$ $\quad n=0$ constant

$\sin(2\pi n x)$ $\quad 1$

<u>Goal</u>: $f(x) \approx c_0 + c_1 e^{2\pi i x} + c_2 e^{4\pi i x} + \cdots + c_n e^{2\pi i n x} + \cdots$

periodic signal

frequency components

$+ c_{-1} e^{-2\pi i x} + c_{-2} e^{-4\pi i x} + \cdots$

How to find the c's?