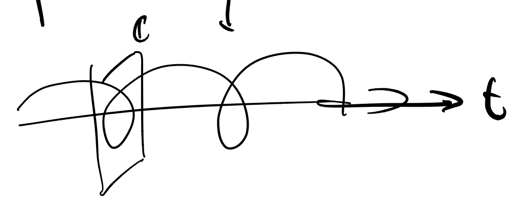
# Complex exponential

$$f(t) = e^{2\pi i k t}$$

periodic oscillating graph

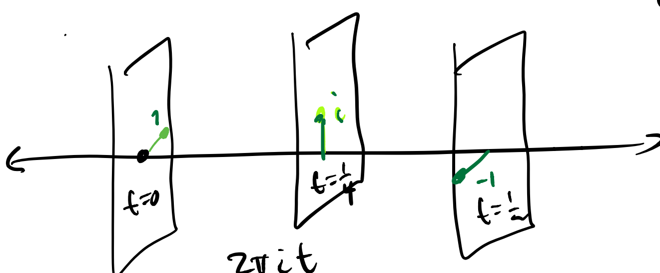



has a period of $\frac{1}{k}$

$$f(t) = c e^{2\pi i t}$$

$c$ = complex number.

$c = re^{i\theta}$



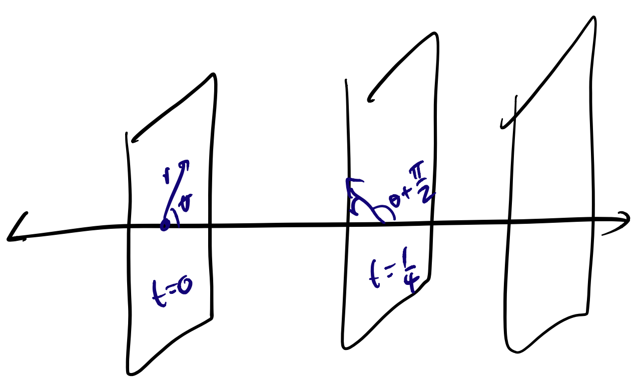

• $e^{2\pi i t}$

• $ce^{2\pi i t} = re^{i\theta} e^{2\pi i t}$

$= re^{(2\pi t + \theta) i}$

amplitude multiply by $r$

angle increased by $\theta$ radians



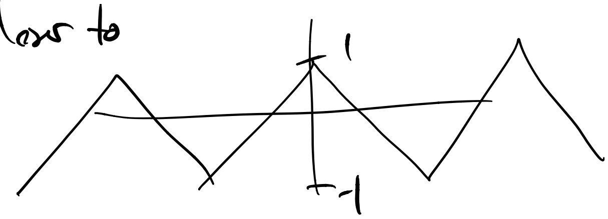
# Classical Fourier Transform

**Example:** (unjustified)

$$\frac{2}{\pi^2}\cos(4\pi t) + \frac{2}{4\pi^2}\cos(8\pi t) + \frac{2}{9\pi^2}\cos(12\pi t)$$

$$+ \frac{2}{16\pi^2}\cos(16\pi t) + \frac{2}{25\pi^2}\cos(20\pi t) + \cdots$$

gets closer & closer to




$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sum_{n>0} \frac{2}{\pi^2 n^2}\cos(4\pi n t) = \sum_{n>0} \frac{2}{\pi^2 n^2}\,\frac{1}{2}\left(e^{4\pi i n t} + e^{-4\pi i n t}\right)$$

$$= \boxed{\sum_{n\neq 0} \frac{1}{\pi^2 n^2} e^{4\pi i n t}}$$

**Main property:**

$$\int_0^1 e^{2\pi i n t}\,dt = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$$

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n t}$$

if we assume $\exists$ such a representation

$f(t)$: signal periodic w/ period 1

$e^{2\pi i n t}$: period $\frac{1}{n}$

how to find $c_k$?

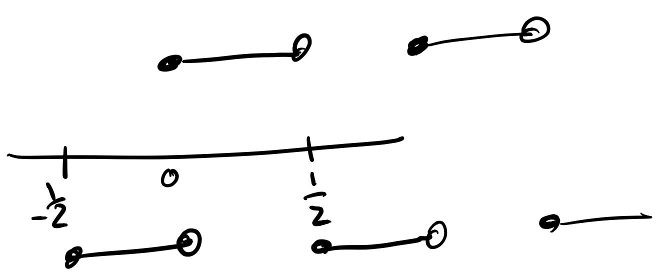
answer is: 

$$f(t)\cdot e^{-2\pi i k t} = \sum c_n e^{2\pi i n t - 2\pi i k t} = \sum c_n e^{2\pi i (n-k)t}$$

$$\int_0^1 f(t) e^{-2\pi i k t}\, dt = \int_0^1 \sum_{n\in\mathbb{Z}} c_n e^{2\pi i (n-k)t}$$

$$= \sum_{n\in\mathbb{Z}} c_n \underbrace{\int_0^1 e^{2\pi i (n-k)t}\, dt}_{\substack{0 \text{ if } n-k\neq 0 \ (n\neq k) \\ 1 \text{ if } n-k=0 \ (n=k)}} = c_k \cdot 1$$

ex:



$$f(t) = c_0 e^{0} + c_1 e^{2\pi i t} + c_{-1} e^{-2\pi i t} + c_2 e^{4\pi i t} + c_{-2} e^{-4\pi i t}$$

$$+ \cdots$$

$$c_0 = \int_0^1 f(t) e^{-0} \partial t = \int_0^{1/2} 1 \partial t + \int_{1/2}^1 (-1) \partial t = 0$$

$$c_1 = \int_0^1 f(t) e^{\pm 2\pi i t} \partial t = \int_0^{1/2} e^{\pm 2\pi i t} \partial t + \int_{1/2}^1 -e^{\pm 2\pi i t} \partial t$$

$c_{-1}$

$$= \pm \frac{1}{2\pi i} \left\{ \left[ e^{\pm 2\pi i t} \right]_0^{1/2} - \left[ e^{\pm 2\pi i t} \right]_{1/2}^1 \right\}$$

$$= \pm \frac{1}{2\pi i} \left\{ \left( e^{\pm i\pi} - e^{0} \right) - \left( e^{\pm 2\pi i} - e^{\pm i\pi} \right) \right\}$$

$$= \pm \frac{1}{2\pi i} \left\{ (-1 - 1) - (1 - (-1)) \right\}$$

$$= \pm \frac{1}{2\pi i} \left\{ -4 \right\} = \frac{2}{i\pi} = -\frac{2i}{\pi}$$

$$= \frac{2i}{\pi}$$

$$0 - \underbrace{\frac{2i}{\pi}}_{c_1} e^{2\pi i t} + \underbrace{\frac{2i}{\pi} e^{-2\pi i t}}_{c_{-1}}$$

# Discrete Fourier Transform

Suppose we have $N$ sample points for our function $f(t)$

periodic.

Convention: $N$ = fixed. instead of thinking about interval

$[0,1]$ & sample pts $0, \frac{1}{n}, \frac{2}{n}, \ldots$

instead, use $[0,N]$ sample pts $0,1,2,\ldots,N-1$

$N$ = sampling rate

Think about sampled function (which is a function on $\mathbb{Z}$)

as a function on the set $\mathbb{Z}/N\mathbb{Z}$

"modular numbers"

"integers modulo $N$"

elements of $\mathbb{Z}/n\mathbb{Z}$ are called $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{N-1}$

where these are shorthands for infinite sets of numbers

$$\overline{0} = \{0, N, -N, 2N, -2N, 3N, \ldots\}$$
$$\overline{1} = \{1, N+1, -N+1, 2N+1, \ldots\}$$

So – our sampled function $[f(0), f(1), \ldots, f(N-1)]^t$

can also think of as $[f(\bar{0}), f(\bar{1}) \ldots]^t$

start w/ $f: \mathbb{R} \to \mathbb{R}$ period $N$, get sampled fcn

$$f^{\text{sampled}}: \mathbb{Z}/N\mathbb{Z} \to \mathbb{R}$$

Start w/ $f$ periodic period $N$, write in terms of

$e^{2\pi i k t/N}$, only sampling at pts $t = 0, 1, \ldots, N-1$

Some $k$'s don't help!

$$k+N \rightsquigarrow e^{2\pi i (k+N) t/N} = e^{2\pi i k t/N} \underbrace{e^{2\pi i N t/N}}_{1}$$

$t = 0, 1, \ldots, N-1 \longrightarrow 1$

there is no way to distinguish wave w/ $k$ vs $k+N$

via sampling at $0, 1, 2, \ldots$

$k = N+1$ $\quad e^{2\pi i N t/N} = e^{2\pi i}$