(Still 1.8)

<u>Definition</u> A Fourier series on $[0, 2\pi]$ is an expression of the form $f(t) = \sum_{n \in \mathbb{Z}} c_n \phi_n(t)$ which is convergent ($\sum |c_n|^2$ converges)

where $\phi_n(t) = e^{int}$

$$\left[\begin{array}{l} \text{on } [0,1] \rightsquigarrow \phi_n = e^{2\pi i n t} \\ \text{on } [0,N] \rightsquigarrow \phi_n = e^{(2\pi/N) i n t} \end{array}\right]$$

to get coeff c_k, $c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \underbrace{\phi_{-k}(t)}_{e^{-ikt}} \, dt$

today's thing: getting coeff c_k as the coord of a vector representation of $f(t)$ via (orth.) projection

$$c_k = \text{"} \langle f(t), \phi_k \rangle \text{"}$$

Hermitian (aka sesquilinear) inner product.

Def A Hermitian inner product on a complex vector space

V is a map $V \times V \longrightarrow \mathbb{C}$

$v, w \longmapsto \langle v, w \rangle$

[want $\langle v, v \rangle = \|v\|^2$ real]

such that ~~want~~ linearity $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$

and $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$

$\lambda \in \mathbb{C}$

and $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$

$\langle \mu v, w \rangle = \bar{\mu} \langle v, w \rangle$

ex: $\mathbb{C}^n = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

$\langle v, v \rangle \leftrightsquigarrow \bar{z} z$

$\langle v, w \rangle \longleftrightarrow \bar{v} \cdot w$

$\langle \vec{a}, \vec{b} \rangle = \sum \bar{a}_i b_i$

$\boxed{\text{and} \quad \langle v, w \rangle = \overline{\langle w, v \rangle}}$

Def $v \perp w \iff \langle v, w \rangle = 0$

Basis $e_1, \dots, e_N$ is orthonormal if $\langle e_i, e_j \rangle = \delta_{ij}$

$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

" is orthogonal if

$\langle e_i, e_j \rangle = 0$ unless $i = j$

Def $\langle , \rangle$ is <u>nondegenerate</u> if $\langle v,v\rangle = 0$ only when $v=0$

$\langle , \rangle$ is <u>positive definite</u> if $\langle v,v\rangle > 0$ all $v \neq 0$

Define a Hermitian inner product on functions on $[0, 2\pi)$ (or on periodic functions w/ period 2π)

via: $\langle f,g\rangle = \int_0^{2\pi} \overline{f(t)}\, g(t)\, dt$

note $\|f\|^2 = \langle f,f\rangle = \int_0^{2\pi} \|f(t)\|^2\, dt$

<u>note:</u> $\langle \phi_n, \phi_m\rangle = \int_0^{2\pi} \overline{\phi_n(t)}\, \phi_m(t)\, dt$

$\phi_n = e^{int}$ $\quad = \int_0^{2\pi} e^{-int} e^{imt}\, dt$

these are orthagonal to each other! $\quad = \int_0^{2\pi} e^{i(m-n)t}\, dt$ $= \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$

if $f(t) = \sum c_n \phi_n(t)$

$\langle \phi_m(t), f(t)\rangle = \langle \phi_m, \sum c_n \phi_n\rangle$

$$= \sum \langle \phi_m, c_n \phi_n \rangle$$

$$= \sum c_n \langle \phi_m, \phi_n \rangle$$

$$= c_m \langle \phi_m, \phi_m \rangle$$

$$= 2\pi c_m$$

$$c_m = \frac{1}{2\pi} \langle \phi_m(t), f(t) \rangle$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \overline{\phi_m(t)}\, f(t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-imt} f(t)\, dt.$$

Back to discrete transform & sampling

Input signal: $f(t)$, assume periodic, sample at N points along the period

$t_0, t_1, t_2, \ldots, t_{N-1}$ equally spaced in period

Convenience: rescale so that $t_0 = 0$, $t_1 = 1$, $t_2 = 2, \ldots$ $t_{N-1} = N-1$

signal period in $[0, N)$

What frequences are we looking for?

From prior discussion: consider $\phi_n = e^{(2\pi/N)nit}$

$$= \Big(\underbrace{\left(e^{2\pi i/N}\right)}^{n}\Big)^{t}$$

useful number for sampling rate N

$\omega = e^{2\pi i/N}$

$$\phi_n = (\omega^n)^t = \omega^{nt}$$

main property of ω is $\omega^N = 1$

if we are only sampling at $t = 0, 1, 2, \ldots, N-1$

then ϕ_n & ϕ_{n+N} look same!

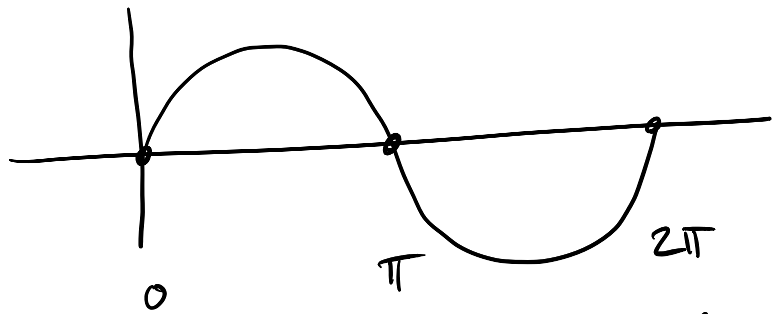
$$\phi_{n+N}(t = 0, 1, 2, \ldots) = \omega^{(n+N)t} = \omega^{nt+Nt} = \omega^{nt}\big(\underbrace{\omega^N}_{1}\big)^t$$

$$= \omega^{nt} = \phi_n(t)$$

<u>Moral</u>: if you are trying to sample a high frequency signal, your sample rate needs to be big as well.

"aliasing"

Nyquist criterion: sample rate must be more than twice the highest frequency you are looking for.



sampling exactly twice as fast – not sufficient.

But – more than twice is always enough.

Since ϕ_N same sampled as ϕ_0

ϕ_{N+1} - - - ϕ_1

when we are trying to "reconstruct" a sampled function $f(t)$ from ϕ_n's, it makes sense to only consider

$$\phi_0, \phi_1, \ldots, \phi_{N-1}$$

Idea: approximate $c_n = \langle \phi_n, f(t) \rangle$

$$= \frac{1}{N}\int_0^N \overline{\phi_n(t)}\, f(t)\, dt$$

$$\text{as} \quad \underbrace{\frac{1}{N}\sum_{m=0}^{N-1} \overline{\phi_n(m)}\, f(m)}_{= \frac{1}{N}\langle \phi_n^{\text{sample}}, f^{\text{sample}}\rangle} \leftarrow (m \text{ stands in for } t)$$

for standard inner product

Does this work?

Sampled function: $f(0), f(1), \ldots, f(N-1)$

we think of as a column vector

$$\begin{bmatrix} f(0) \\ \vdots \\ f(N-1) \end{bmatrix} \in \mathbb{C}^N$$

in particular, $\phi_n(t) = \omega^{nt} \underset{\text{sample}}{\rightsquigarrow} \begin{bmatrix} \omega^{n\cdot 0} \\ \omega^{n\cdot 1} \\ \vdots \\ \omega^{n(N-1)} \end{bmatrix} \in \mathbb{C}^N$

standard inner product on $\mathbb{C}^n$:

$$\left\langle \begin{bmatrix} v_0 \\ \vdots \\ v_{N-1} \end{bmatrix}, \begin{bmatrix} w_0 \\ \vdots \\ w_{N-1} \end{bmatrix} \right\rangle \equiv \sum_{n=0}^{N-1} \overline{v_n}\, w_n$$

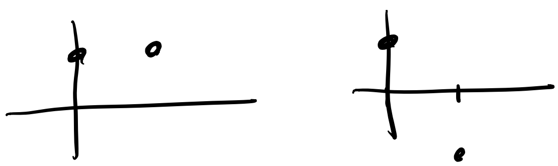
Notation: Sampled basic waveforms

$$\phi_n(t)^{\text{sampled}} \equiv E_n = \begin{bmatrix} \omega^{n0} \\ \omega^{n1} \\ \vdots \\ \omega^{n(N-1)} \end{bmatrix}$$

$N=2 \qquad \phi_0, \phi_1 = \left(e^{2\pi i/2}\right)^{1\cdot t} \qquad t=0,1$

$$\phi_0 = \left(e^{2\pi i/2}\right)^{0\cdot t} = 1 \rightsquigarrow E_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{array}{l} \leftarrow t=0 \\ \leftarrow t=1 \end{array}$$

$$\left(e^{2\pi i/2}\right)^{1\cdot t} = (-1)^t \rightsquigarrow E_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{array}{l} \leftarrow t=0 \\ \leftarrow t=1 \end{array}$$



$N=4 \qquad \phi_0, \phi_1, \phi_2, \phi_3 \qquad \left(e^{2\pi i/4}\right)^{nt} \qquad n=0,1,2,3 \quad t=0,1,2,3$

$$e^{2\pi i/4} = e^{\pi/2\, i} = i$$

$$(i)^{nt}$$

$$n=0 \quad E_0 = \begin{bmatrix} i^{0\cdot 0} \\ i^{0\cdot 1} \\ i^{0\cdot 2} \\ i^{0\cdot 3} \end{bmatrix} \quad E_1 = \begin{bmatrix} i^{1\cdot 0} \\ i^{1\cdot 1} \\ i^{1\cdot 2} \\ i^{1\cdot 3} \end{bmatrix} \quad E_2 = \begin{bmatrix} i^{2\cdot 0} \\ i^{2\cdot 1} \\ \vdots \end{bmatrix}$$

$$E_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

<u>Fact:</u> $\langle E_n, E_m \rangle = N \delta_{n,m}$!

So, can determine c_n's by inner product.

Suppose have a sampled signal $f_0, \dots, f_{N-1} \rightsquigarrow \vec{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$

Suppose can write it $\vec{f} = \sum_{n=0}^{N-1} c_n E_n$

then: $c_m = \frac{1}{N} \langle E_m, \vec{f} \rangle$!