Goal 1: Discrete Fourier Transform

Basic idea:

given our sampled function $f(t) \rightsquigarrow f[j]$

$j \in \{0, \dots, N-1\}$

$\mathbb{Z}/N\mathbb{Z}$

$f[j] \in \ell_{\mathbb{C}}[\mathbb{Z}/N\mathbb{Z}]$ or $\mathbb{C}^N$

"complex valued functions on the set $\mathbb{Z}/N\mathbb{Z}$"

Basic functions/vectors : $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← j^{th} position

$$e_j[k] = \delta_{jk}$$

Basic waveforms E_j : $E_j[k] = \omega^{jk}$

$\omega = e^{2\pi i/N}$

"primitive N^{th} root of unity"

$$E_j = \begin{bmatrix} \omega^0 \\ \omega^j \\ \omega^{2j} \\ \vdots \\ \omega^{(N-1)j} \end{bmatrix}$$

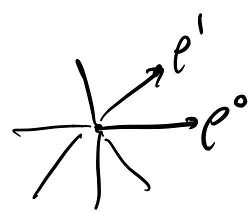
Main observation (Prop 2.1)

$\langle E_j, E_k \rangle = N$ if $j = k$, 0 else.

Prove this:

Notice that if ρ is an M^{th} root of unity ($\rho \neq 1$) $\rho^M = 1$

then $\rho^0 + \rho^1 + \cdots + \rho^{M-1} = 0$



Since: $(\rho^0 + \rho^1 + \cdots + \rho^{M-1})(1-\rho)$

$= \rho^0 + \rho^1 + \cdots + \rho^{M-1} - \rho^1 - \rho^2 - \cdots - \rho^M$

$= \rho^0 - \rho^M = 1 - 1 = 0$

Since $\rho \neq 1$, the other factor: $\rho^0 + \cdots + \rho^{M-1}$ must be 0.

$$\langle E_j, E_k \rangle = \sum_{l=0}^{N-1} \overline{E_j[l]}\, E_k[l]$$ "Hermitian inner product"

$$= \sum_{l=0}^{N-1} \overline{\omega^{jl}}\, \omega^{kl}$$ $\omega = e^{2\pi i/N}$

$\overline{\omega} = e^{-2\pi i/N}$

$$= \sum_{l=0}^{N-1} (\overline{\omega})^{jl}\, \omega^{kl}$$ $\omega\overline{\omega} = e^0 = 1$

$\overline{\omega} = \omega^{-1}$

$$= \sum_{\ell=0}^{N-1} \omega^{-j\ell+k\ell} = \sum_{\ell=0}^{N-1} (\omega^{k-j})^{\ell}$$

So: ω^{k-j} is some root of unity, call it ρ.

say its an M^{th} root of unity. $\rightsquigarrow N = dM$

$$(\rho^0 + \rho^1 + \cdots + \rho^{M-1}) + (\rho^M + \rho^{M+1} + \cdots + \rho^{2M-1}) + (\rho^{2M} + \cdots$$

$$+ (\cdots \underset{\underset{\rho^{N-1}}{\shortparallel}}{\rho^{dM-1}})$$

$$= \underbrace{(\rho^0 + \cdots + \rho^{M-1})}_{0} + \rho^M \underbrace{(\rho^0 + \cdots + \rho^{M-1})}_{0} + \rho^{2M} \underbrace{(\cdots)}_{0} \cdots$$

$$= 0$$

$\underline{\text{if}}$ $M \neq 1$

So get 0 unless $k = j$.

if $k = j$, get $\langle E_j, E_k \rangle = \cdots = \sum_{\ell=0}^{N-1} (\omega^{j-k})^{\ell}$

$$= \sum_{\ell=0}^{N-1} 1^{\ell} = N.$$

So: E_j basis is an orthogonal basis

$u_j \equiv \frac{1}{\sqrt{N}} E_j$ is an orthonormal basis

$$\langle u_j, u_k \rangle = \delta_{jk}$$

$$\langle \tfrac{1}{\sqrt{N}} E_j, \tfrac{1}{\sqrt{N}} E_k \rangle = \frac{\bar{1}}{\sqrt{N}} \frac{1}{\sqrt{N}} \langle E_j, E_k \rangle$$

$$= \frac{1}{N} N \cdot \delta_{jk} = \delta_{jk}$$

Why is $\{E_j\}$ actually a basis?

Note: there are the correct number: $E_0, \dots, E_{N-1}$

and they are linearly independent, since

$$\sum_{j=0}^{N-1} a_j E_j = \vec{0} \quad \text{then}, \ \langle E_k, \ \rangle \text{ each side}$$

for each k

$$\langle E_k, \sum a_j E_j \rangle = \langle E_k, \vec{0} \rangle = 0$$

$$a_j \langle E_k, \sum_j E_j \rangle = \sum_j N a_j \delta_{jk} = 0$$

$$= N a_k = 0$$

$$\Rightarrow a_k = 0 \text{ all } k \ \square$$

Given $f[j]$ sampled function

Know: since E_k's are a basis, can write

$$f = \sum a_k E_k$$ since this is true, can solve by $\langle E_j, \ \rangle$ both sides.

$$\langle E_j, f \rangle = \sum_k a_k \langle E_j, E_k \rangle$$

$$= \sum_k a_k \delta_{jk} N = a_j N$$

$$a_j = \frac{1}{N} \langle E_j, f \rangle = \frac{1}{N} \sum_{k=0}^{N-1} \overline{E_j[k]} f[k]$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \omega^{-jk} f[k]$$

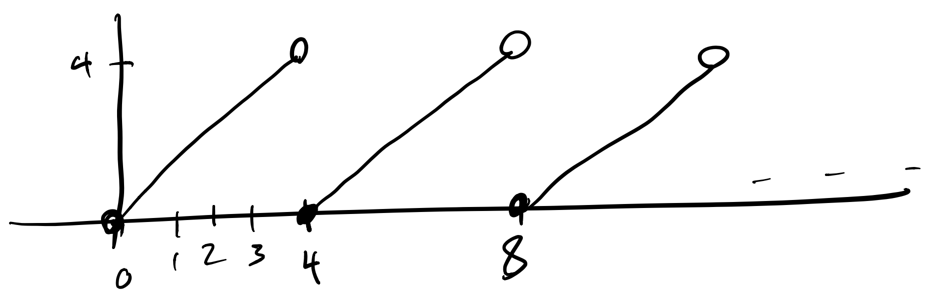
$$[f]_{\{e_j\}} = \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[N-1] \end{bmatrix} \quad f = \sum f[j] e_j \qquad [f]_{\{E_j\}} = \begin{bmatrix} \hat{f}[0] \\ \hat{f}[1] \\ \vdots \\ \hat{f}[N-1] \end{bmatrix}$$

$$f = \sum a_j E_j$$

Notation: $\hat{f}[j] = a_j$

Ex: $N = 4$ $\omega = i$

$f(t) = t$ $[0,4)$ extended periodically



$$[f]_{\{e_i\}} = \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[3] \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$\hat{f}[0] = \text{coeff of } \underset{\omega^{0k}}{E_0}$ $\hat{f}[j] = \frac{1}{N} \langle E_j, f \rangle$

$\hat{f}[j] = a_j = \frac{1}{N} \sum_k \omega^{-jk} f[k]$

$j = 0$ $\frac{1}{4} \sum_{k=0}^{3} \omega^0 f[k] = 3/2$

$= \frac{1}{4}\left(f(0) + f(1) + f(2) + f(3)\right)$

$\hat{f}[1] = \text{coeff } E_1$ $\omega^{1 \cdot k}$

$= \frac{1}{4} \sum_{k=0}^{3} \omega^{-k} f[k] = \frac{1}{4}\Big(i^{-0} f(0) + i^{-1} f(1)$

$+ i^{-2} f(2) + i^{-3} f(3)\Big)$

$$= \frac{1}{4}\left(f(0) - i f(1) - f(2) + i f(3)\right)$$

$$= \frac{1}{4}(0 - i - 2 + i3)$$

$$= \frac{1}{4}(-2 + 2i) = -\frac{1}{2} + \frac{1}{2} i$$

<u>Alternate interpretation</u>

$$\begin{bmatrix} f[0] \\ \vdots \\ f[N-1] \end{bmatrix} \longrightarrow \begin{bmatrix} \hat{f}[0] \\ \vdots \\ \hat{f}[N-1] \end{bmatrix}$$

is a <u>change of basis</u> given by a change of basis matrix. $\tilde{F}$

matrix whose columns are the new basis in terms of the old one.

opposite change is easy to write conceptually:

$[E_j]_{\{e_k\}}$ ← vector in $\mathbb{C}^N$ whose ℓ^{th}-coordinate is $E_j[\ell]$ $= \omega^{j\ell}$

$$(E_j)_{\{e_k\}} = \begin{bmatrix} \omega^{j\cdot 0} \\ \omega^{j1} \\ \vdots \\ \omega^{j(N-1)} \end{bmatrix}$$

C. of B matrix from E's to e's has for its j^{th} column: E_j in basis of e's

matrix is:

$$G = \tilde{F}^{-1} = \begin{bmatrix} \omega^{00} & \omega^{1\cdot 0} & \cdots & \omega^{(N-1)0} \\ \omega^{01} & \omega^{1\cdot 1} & & \omega^{(N-1)1} \\ \omega^{02} & \omega^{1\cdot 2} & & \vdots \\ \vdots & \vdots & & \\ \omega^{0(N-1)} & \omega^{1(N-1)} & & \omega^{(N-1)(N-1)} \end{bmatrix}$$

j,k entry is $\omega^{(j-1)(k-1)}$

$\overline{G}G$ ← j,k entry is the inner prod of E_{j-1} & $\overline{E}_{k-1}$

$$\overline{G}G = N \cdot I_N$$

$$\langle E_{j-1}, E_{k-1} \rangle = \delta_{j-1,k-1} N = \delta_{j,k} N$$

$\Rightarrow \overline{G}G = N I_N$

$\frac{1}{N}\overline{G} = G^{-1} = \tilde{F}$ ← change of basis from standard basis to waveforms E's

$$\tilde{F} = \frac{1}{N}\left(\qquad \right)$$

matrix whose jk entry is $\omega^{-jk} = (\overline{\omega})^{jk}$

"F"

ex: $N=2 \quad \omega=-1 \quad \overline{\omega}=-1$

$$F_2 = \begin{bmatrix} (-1)^{0\cdot 0} & (-1)^{1,0} \\ (-1)^{0\cdot 1} & (-1)^{1\cdot 1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \tilde{F}_2 = \frac{1}{2}F_2$$

powers of 1 | powers of $-i$ | powers of $(-i)^2 = -1$ | powers of $(-i)^3 = i$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \qquad \tilde{F}_4 = \frac{1}{4}F_4$$

$f(t) = t \quad [0,4)$

$$F_4 f = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ \vdots \\ \vdots \end{bmatrix}$$

Practice: $f(t) = 1 - x^2$ on $[0,4)$ continued periodically

$\hat{f}[2]$

$$f(t) = \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \end{bmatrix} = 1\,e_0 + 0\,e_1 - 3\,e_2 - 8\,e_3$$

$$\frac{1}{4}\langle E_2, f\rangle = \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -3 \\ -8 \end{bmatrix} = \frac{1}{4}\,6 = \frac{3}{2}$$