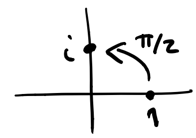
$$S = \begin{cases} e_0 \to e_1 \\ e_1 \to e_2 \\ \vdots \\ e_3 \to e_0 \end{cases} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$E_j = \begin{bmatrix} \omega^{j\cdot 0} \\ \omega^{j\cdot 1} \\ \vdots \\ \omega^{j\cdot(N-1)} \end{bmatrix} = \begin{bmatrix} i^{j\cdot 0} \\ i^{j\cdot 1} \\ i^{j\cdot 2} \\ i^{j\cdot 3} \end{bmatrix}$$

$$j=0 \quad E_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$j=1 \Rightarrow E_1 \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}$$

$$N=4 \quad \omega = e^{2\pi i/N} = e^{2\pi i/4} = e^{i\pi/2}$$



$$\omega = i$$

$$SE_0 = S\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = E_0$$

$$SE_0 = 1\cdot E_0$$

E_0 is an Eigenvectr w/ Eigenvalue 1.

$$SE_1 = S\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} -i \\ 1 \\ i \\ -1 \end{bmatrix} = (-i)\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = (-i)E_1$$

E_1 is an Eigenvectr w/ value $-i$

General picture: (General N)

$$SE_j = S\begin{bmatrix} \omega^{j\cdot 0} \\ \omega^{j\cdot 1} \\ \vdots \\ \omega^{j(N-1)} \end{bmatrix} = \begin{bmatrix} \omega^{j(N-1)} \\ \omega^{j\cdot 0} \\ \omega^{j\cdot 1} \\ \vdots \\ \omega^{j(N-2)} \end{bmatrix} = \omega^{j(N-1)} \begin{bmatrix} \omega^{j\cdot 0} \\ \omega^{j\cdot 1} \\ \omega^{j\cdot 2} \\ \vdots \\ \omega^{j(N-1)} \end{bmatrix}$$

$\omega^N = 1$

$$\omega^{j(N-1)} = \omega^{jN-j} = (\omega^N)^j \omega^{-j} = \omega^{-j}$$

So: $SE_j = \omega^{-j} E_j$

so E_j is an E. vect. w/ value ω^{-j}

Recall: $\mathbb{C}^N = \ell_{\mathbb{C}}[\mathbb{Z}/N\mathbb{Z}]$

have basis vectors $e_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← jth place

$e_j[k] = \delta_{jk}$

$$E_j = \begin{bmatrix} \omega^{j\cdot 0} \\ \vdots \\ \omega^{j\cdot(N-1)} \end{bmatrix}$$

$\mathbb{C}^N$ has an inner product

$$\left\langle \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix}, \begin{bmatrix} b_0 \\ \vdots \\ b_{N-1} \end{bmatrix} \right\rangle = \sum_{j=0}^{N-1} \overline{a_j} b_j$$

$$\langle E_j, E_k \rangle = \delta_{jk} \cdot N$$

matrix G = change of basis from E's to e's

$$G = \begin{bmatrix} E_0 \,|\, E_1 \,|\, \cdots \, E_{N-1} \end{bmatrix} \quad \text{then} \quad \overline{G}G = N I_N$$

So $\Rightarrow$ $\frac{1}{N}\overline{G} = G^{-1}$ is the change of basis from e_j's to E_j's

$$\overline{G} = F$$

So $G^{-1} = \frac{1}{N} F$

"Fourier Matrix"

given $y = \begin{bmatrix} y_0 \\ \vdots \\ y_{N-1} \end{bmatrix}$ sampled fcn. $\quad \frac{1}{N} F y = \begin{bmatrix} \hat{y}_0 \\ \vdots \\ \hat{y}_{N-1} \end{bmatrix}$

$$F = \begin{bmatrix} \overline{E_0} \,|\, \overline{E_1} \,|\, \cdots \,|\, \overline{E_{N-1}} \end{bmatrix}$$

$\hat{y}_j$'s are coef of j^{th} waveform E_j

$$y = \sum y_j e_j = \sum \hat{y}_j E_j$$

Filters (& convolutions)

<u>Idea:</u> $f(t) \rightsquigarrow$ new signal

various frequencies

accentuate some frequencies, remove/decrease others

In Fourier basis, this is an obvious thing to do: these are diagonal!

$E_0, E_1, E_2, \dots, E_{N-1}$ want to increase E_j by a factor of 2 & dec E_k by a factor of 3 & keep others same:

$$\begin{array}{cc} \begin{bmatrix} 1 & & & & & & & & & \\ & \ddots & & & & & & 0 & & \\ & & 1 & & & & & & & \\ & & & 2 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & & & & & \left(\frac{1}{3}\right) & & \\ & 0 & & & & & & & 1 & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ j^{th} \\ \\ \\ \\ k^{th} \\ \\ \\ \\ \end{array} \end{array}$$

Can think about this Diag. matrix as "pointwise mult." by a sampled fcn

$$g = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 2 \\ 1 \\ \vdots \\ 1 \\ 1/3 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$f[\ell]\, g[\ell] = fg[\ell]$

in wave basis!

Notation / Convention: y - signal

f - a function, use to create a filter.

Idea: f has various frequencies occur.

$f = \sum_s \hat{f}[k] E_k$ filter other signals based on proportions waveforms arise in f.

concretely: write $f * y$

"y filtered via f"

$$\widehat{f*y}[k] = \hat{f}[k]\,\hat{y}[k]$$

$$f * y = \sum \hat{f}[k]\,\hat{y}[k]\, E_k$$

As we've observed:

S - shift oprator has E_k's as eignvectors gives a set of N distinct e.vects w/ distinct, nonzero e.vals.

Theorem (Linear Algebra) If T is a nonsigular diagonalizable matrix with distinct nonzero e.vals then the only matrices which commute with it are linr combinations of powers of T.

(T' commutes w/ T $\Leftrightarrow$ $T'T = TT'$)

$\Rightarrow$ the only matrices which commute with S are linr. combs of powrs of S.

Def A circulant matrix is one of the form

$$T = \sum_{j=0}^{N-1} a_j S^j$$

$$3 + 2S - S^3$$

<u>Note:</u> in the wave basis, S is diagonal as are all S^k's & all circulant matrices are diagonal.

But also the lin transformation $y \xmapsto{T_f} f * y$ is also diagonal in wave basis

$$\begin{bmatrix} \hat{f}[0] & & & 0 \\ & \hat{f}[1] & & \\ & & \ddots & \\ 0 & & & \hat{f}[N-1] \end{bmatrix}$$

$\Rightarrow$ T_f commutes w/ S. But then $\Rightarrow$ T_f is <u>circulant</u>

In fact,

$$\left(\begin{array}{c}\text{diag. matrices in} \\ \text{wave basis} \\ \text{is } N\text{-dim'l v.space}\end{array}\right) \subset \left\{\begin{array}{c}\text{matrices which} \\ \text{comm. w/ } S\end{array}\right\}$$

$$\cap$$

$$\supset \left\{\begin{array}{c}\text{circulant} \\ \text{matrices}\end{array}\right\}$$

$$\| $$

filters $\begin{bmatrix} \hat{f}[0] & & & 0 \\ & \hat{f}[1] & & \\ 0 & & \ddots & \hat{f}[N-1] \end{bmatrix}$

<u>Def</u> if f & g are sampled fns (in $\ell_{\mathbb{C}}[\mathbb{Z}/N\mathbb{Z}]$ $\cong \mathbb{C}^N$)

we define $f \star g$ as follows

$$e_j \star e_k = e_{j+k} \quad \text{extend:}$$

$$f = \sum f[j] e_j \qquad g = \sum g[k] e_k$$

$$f \star g = \sum_{j,k} f[j] g[k] e_j \star e_k$$

$$= \sum_{j,k} f[j] g[k] e_{j+k} = \sum_{\ell} \Big(\sum_{j+k = \ell} f[j] g[k] \Big) e_\ell$$

$$= \sum_{\ell} \sum_{j} f[j] g[\ell - j] e_\ell$$

$$(k = \ell - j)$$

<u>Prop</u>: $f * g = f \star g$

<u>Pf</u>: $E_j E_k = E_{j+k}$

$$F^{-1}(E_j E_k) = F^{-1}(E_{j+k}) =$$