Fourier Transform

f signal

$$f = \sum_{k=0}^{N-1} f[k] e_k = \sum_{k=0}^{N-1} \frac{\hat{f}[k]}{N} E_k$$

$\hat{f}[k]$ = Fourier Coefficients of f

$$F_N = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \cdots & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^{-1} & \omega^{-2} & & \omega^{-j} & & \vdots \\ \omega^0 & \omega^{-2} & \omega^{-4} & & \omega^{-2j} & & \vdots \\ \vdots & \vdots & \vdots & & \vdots & & \omega^{-(N-1)(N-1)} \\ \omega^0 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \cdots & \omega^{-j(N-1)} & \cdots & \end{bmatrix}$$

$$F_N \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[N-1] \end{bmatrix} = \begin{bmatrix} \hat{f}[0] \\ \vdots \\ \hat{f}[N-1] \end{bmatrix}$$

Linear trans assoc. to F_N is called the "Discrete Fourier Transform" DFT

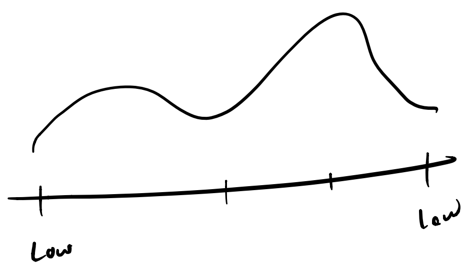
$$G_N = \begin{bmatrix} \omega^0 & \omega^0 & \cdots & \omega^0 \\ \omega^0 & \omega^1 & & \omega^{N-1} \\ \vdots & \vdots & & \omega^{2(N-1)} \\ \vdots & \vdots & & \vdots \\ \omega^0 & \omega^{N-1} & \cdots & \omega^{(N-1)(N-1)} \end{bmatrix}$$

$$F_N G_N = N I_N$$

$$\boxed{\hat{f}[k] = \sum_{j=0}^{N-1} \omega^{-jk} f[j]}$$

Filters

$f * y$ "filter from f, applied to y"



$$\boxed{\widehat{f*y}[k] = \hat{f}[k]\,\hat{y}[k]}$$

$$\boxed{e_k * e_j = e_{k+j}}$$

Circular "Convolution"

Def $\quad f * y[j] = \sum_{k=0}^{N-1} f[k]\, y[j-k]$

(Def 2.4 text)

Circulant Matrices

S "shift" linear transformation taking $\ell_{\mathbb{C}}[\mathbb{Z}/N\mathbb{Z}]$ to itself.

$(Sf)[k] = f[k-1]$

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & \vdots & 0 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & 0 & & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$Se_k = e_{k+1}$

<u>Def</u> A circulant matrix is one of the form

$$\sum_{j=0}^{N-1} a_j S^j \quad \text{some } a_j\text{'s in } \mathbb{C}.$$

<u>Ex:</u> $T_2 = \frac{1}{2}(I_N + S)$ ⟵ "moving average"

"smoothes out a signal"

dampen high frequencies.

$$T_5 = \frac{1}{5}(I_N + S + S^2 + S^3 + S^4)$$

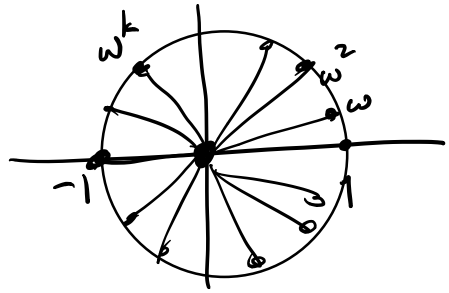
How does T_2 act on waveforms?

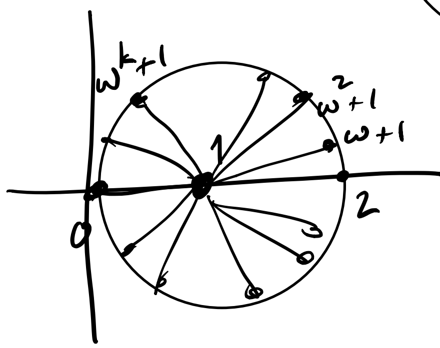
$$SE_k = S\begin{bmatrix} \omega^0 \\ \omega^k \\ \omega^{2k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix} = \begin{bmatrix} \omega^{(N-1)k} \\ \omega^0 \\ \omega^k \\ \vdots \\ \omega^{(N-2)k} \end{bmatrix}$$

(note: $\omega^{(N-1)k} = \omega^{Nk-k} = \underbrace{\omega^{Nk}}_{1}\omega^{-k} = \omega^{-k}$

$$= \omega^{-k}\begin{bmatrix} \omega^0 \\ \omega^k \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

$$\frac{1}{2}(I+S)E_k = \frac{1}{2}E_k + \frac{\omega^{-k}}{2}E_k = \underbrace{\left(\frac{1}{2}(1+\omega^{-k})\right)}_{?}E_k$$





waveforms w/ $k \sim N/2$ are mult. by something close to 0.
waveforms w/ k small are left alone.

$$E_k \longmapsto \frac{1}{2}(1+\omega^{-k})E_k$$

$$f = \sum_{k=0}^{N} \frac{1}{2N}(1+\omega^{-k})E_k \qquad \hat{f}[k] = \frac{1}{2}(1+\omega^{-k})$$

$$y \longmapsto f * y$$

END of "REVIEW"

FAST FOURIER TRANSFORM

$$\hat{f}[j] = \sum_{k=0}^{M-1} \omega^{-jk} f[k]$$

assume $N = 2M$

ω = Nth root of unity

$\omega^2 = \varsigma = M^{th}$ root of unity

$(\omega^2)^M = \omega^{2M} = \omega^M = 1$

"Downsampling"

$$f \longleftrightarrow \begin{bmatrix} f[0] \\ f[1] \\ \vdots \\ f[M-1] \end{bmatrix} \qquad f_{even} = \begin{bmatrix} f[0] \\ f[2] \\ \vdots \\ f[2M-2] \end{bmatrix} \qquad f_{odd} = \begin{bmatrix} f[1] \\ f[3] \\ \vdots \\ f[2M-1] \end{bmatrix}$$

$m = 2M$

$$\hat{f}[j] = \sum_{k \text{ even}} \omega^{-jk} f[k] + \sum_{k \text{ odd}} \omega^{-jk} f[k]$$

$$= \sum_{k=0}^{M-1} \omega^{-j(2k)} f[2k] + \sum_{k=0}^{M-1} \omega^{-j(2k+1)} f[2k+1]$$

$$= \sum_{k=0}^{M-1} \varsigma^{-jk} f[2k] + \omega^{-j} \sum_{k=0}^{M-1} \varsigma^{-jk} f[2k+1]$$

$$= \underbrace{\sum_{k=0}^{M-1} \varsigma^{-jk} f_{even}[k]}_{\hat{f}_{even}[j]} + \omega^{-j} \underbrace{\sum_{k=0}^{M-1} \varsigma^{-jk} f_{odd}[k]}_{\hat{f}_{odd}[j]}$$

$$\hat{f}[j] = \widehat{f_{even}}[j] + \omega^{-j}\,\widehat{f_{odd}}[j]$$

$$F_N = \begin{bmatrix} F_M & 0 \\ 0 & \omega^{-j} F_M \end{bmatrix} (\text{Perm})$$

$F_1 = [1]$ $E_1 = [1]$ $e_1 = [1]$ $N = 1$

F_2 $f = \begin{bmatrix} f[0] \\ f[1] \end{bmatrix}$ $N = 2$

$\omega = -1$

$$\hat{f}[j] = \widehat{f_{even}}[j] + \omega^{-j}\,\widehat{f_{odd}}[j]$$

$$= \underbrace{\widehat{f_{even}}}_{=f_{even} = [f[0]]}[j] + (-1)^{-j}\,\underbrace{\widehat{f_{odd}}}_{= f_{odd} = f[1]}[j]$$

$$= f[0] + (-1)^{-j} f[1]$$

$\hat{f}[0] = f[0] + f[1]$ $\hat{f}[1] = f[0] - f[1]$

$$\hat{f}[j] = \underbrace{\widehat{f_{even}}[j]}_{F_2 \begin{bmatrix} f[0] \\ f[2] \end{bmatrix}} + (i)^{-j}\,\underbrace{\widehat{f_{odd}}[j]}_{\begin{bmatrix} f[1] \\ f[3] \end{bmatrix}}$$

$N = 4$

$\omega = i$

$$\hat{f}[3] = \underbrace{\hat{f}_{even}[3]}_{\text{periodic w/ period 2}} + (i)^3\,\hat{f}_{odd}[3]$$

$$= \hat{f}_{even}[1] = f[0] - f[2]$$

$$\hat{f}_{odd}[1] = f[1] - f[3]$$

$$\hat{f}[3] = f[0] - f[2] + i\big(f[1] - f[3]\big)$$

$$= f[0] + if[1] - f[2] - if[3]$$