Lecture 15: Chain Rule, Part 3

Tuesday, February 21, 2017 12:32 PM

Logarithmic Differentiation

Sometimes logs make derivatives easier.
(they help simplify products, quotients & exponents)

ex: $$\frac{d}{dx}\ln\left[\frac{(3x+2)^{2x}(x^2-2)}{(3x+7)^2\sqrt{(x+1)}}\right]$$

lots of hard pieces w/out ln

ex: $$\frac{d}{dx}(3x+2)^{2x} = e^{2x\ln(3x+2)}$$

$$= e^{2x\ln(3x+2)}\frac{d}{dx}2x\ln(3x+2)\ldots$$

$$\frac{d}{dx}\ln\left[\frac{(3x+2)^{2x}\cdot(x^2-2)}{(3x+7)^2\sqrt{(x+1)}}\right] = \frac{d}{dx}\left[\ln\left((3x+2)^{2x}(x^2-2)\right) - \ln\left((3x+7)^2\sqrt{x+1}\right)\right]$$

$$= \frac{d}{dx}\left[\ln(3x+2)^{2x} + \ln(x^2-2) - \ln(3x+7)^2 - \ln(x+1)^{1/2}\right]$$

$$= \frac{d}{dx}\left[2x\ln(3x+2) + \ln(x^2-2) - 2\ln(3x+7) - \frac{1}{2}\ln(x+1)\right]$$

$$\frac{d}{dx}\left[\quad \right]$$

$$= \left(2x \cdot \frac{1}{3x+2} \cdot (3) + 2 \cdot \ln(3x+2)\right) + \frac{1}{x^2-2}(2x) - 2\frac{1}{3x+7}(3)$$

$$-\frac{1}{2}\frac{1}{x+1} \cdot 1$$

Idea: given $f(x)$ & want $f'(x)$

know how to find $\frac{d}{dx}\ln f(x)$

Strategy: write $\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} \cdot f'(x)$

$$\frac{d}{dx}\ln f(x) = \text{(something we know)}$$

solve for $f'(x) = f(x) \cdot \text{(something we know)} = f(x) \cdot \frac{d}{dx}\ln f(x)$

ex:

$$\frac{d}{dx}\underbrace{\frac{x^x e^x (1-x)}{(x^2+2)^3\sqrt{x+1}}}_{f(x)}$$

$$\frac{d}{dx}\ln f(x) = \frac{1}{f(x)} f'(x)$$

$$f'(x) = f(x) \cdot \frac{d}{dx}\ln f(x)$$

$$\ln f(x) = x\ln x + x + \ln(1-x) - 3\ln(x^2+2) - \frac{1}{2}\ln(x+1)$$

$$\frac{d}{dx}\ln f(x) = x \cdot \frac{1}{x} + \ln x + 1 + \frac{1}{1-x} \cdot (-1) - 3\frac{1}{x^2+2}(2x) - \frac{1}{2}\frac{1}{x+1}$$

$$f'(x) = \left(\frac{x^x e^x (1-x)}{(x^2+2)^3\sqrt{x+1}}\right)\left[1 + \ln x + 1 - \frac{1}{1-x} - \frac{6x}{x^2+2} - \frac{1}{2(x+1)}\right]$$

$$f'(x) = \left(\frac{x^x e^x (1-x)}{(x^2+2)^3\sqrt{x+1}}\right)\left[1 + \ln x + 1 - \frac{1}{1-x} - \frac{6x}{x^2+2} - \frac{1}{2(x+1)}\right]$$

$$\frac{d}{dx}\underbrace{\frac{(1-x)^2}{x^2+2}}_{f(x)}$$

$$f'(x) = f(x)\cdot\frac{d}{dx}\ln f(x)$$

$$\ln f(x) = 2\ln(1-x) - \ln(x^2+2)$$

$$\frac{d}{dx}\ln f(x) = 2\,\frac{1}{1-x}\cdot(-1) - \frac{1}{x^2+2}\cdot(2x)$$

$$f'(x) = \left(\frac{(1-x)^2}{x^2+2}\right)\left(\frac{-2}{1-x} - \frac{2x}{x^2+2}\right)$$

$$\frac{d}{dx}x^x$$

$$\ln x^x = x\ln x$$

$$\frac{d}{dx}\ln x^x = \frac{d}{dx}x\ln x = x\cdot\frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

$$\frac{d}{dx}x^x = x^x\cdot\frac{d}{dx}\ln x^x = x^x(1+\ln x)$$

$$f'(x) = \frac{d}{dx}f(x) = f(x)\,\frac{d}{dx}\ln f(x)$$

$$\frac{d}{dx}2^x(1+x)^3 = 2^x(1+x)^3\cdot\frac{d}{dx}\ln\left(2^x(1+x)^3\right) = 2^x(1+x)^3\left(\ln 2 + \frac{3}{1+x}\right)$$

$$\ln\left(2^x(1+x)^3\right) = \ln 2^x + \ln(1+x)^3$$

$= x\ln 2 + 3\ln(1+x)$

$$\frac{d}{dx}\ln\left(2^x(1+x)^3\right) = \ln 2 + 3\cdot\frac{1}{1+x}\cdot 1$$

$$= \ln 2 + \frac{3}{1+x}$$

Derivatives of inverse functions

$\frac{d}{dx}\ln x$

$$e^{\ln x} = x$$

$$\frac{d}{dx}e^{\ln x} = \frac{d}{dx}1$$

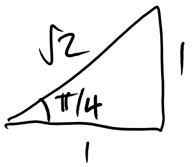
$$= e^{\ln x}\cdot\frac{d}{dx}\ln x = 1$$

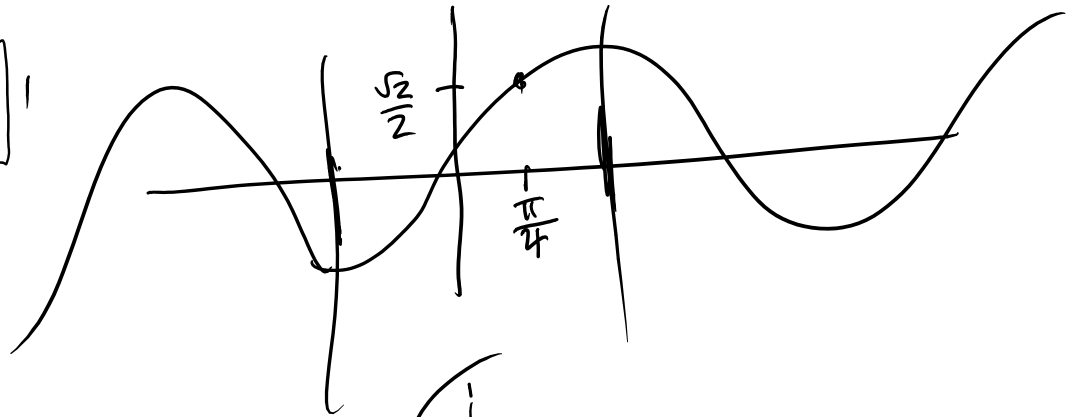
$$\frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

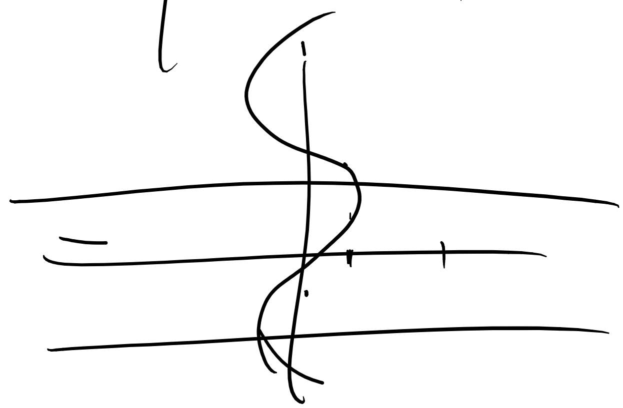
example: arcsin x (AKA $\sin^{-1}x$)

function which takes x and gives an angle between $-\frac{\pi}{2}$ & $\frac{\pi}{2}$ whose sine is x.

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \longrightarrow \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$





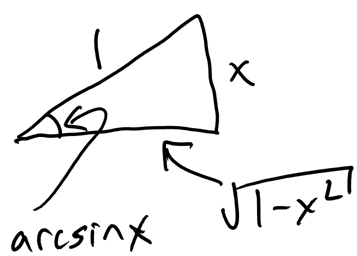


$\sin(\arcsin x) = x$ use this to find $\frac{d}{dx} \arcsin x$

$\frac{d}{dx} \sin(\arcsin x) = \frac{d}{dx} x = 1$ (take derivatives of both sides)

$\cos(\arcsin x) \cdot \frac{d}{dx} \arcsin x = 1$

$$\frac{d}{dx} \arcsin x = \frac{1}{\cos(\arcsin x)}$$



$$= \frac{1}{\sqrt{1-x^2}}$$

$\tan(\arctan x) = x$

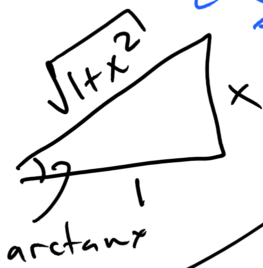
$\frac{d}{dx} \tan(\arctan x) = \frac{d}{dx} x = 1$

$$\sec^2(\arctan x)\cdot\frac{d}{dx}\arctan x = 1$$

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)} = \cos^2(\arctan x)$$

uh-oh. need to figure out cos(arctanx) ...

draw △



solve for cos

$$\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

use in formula.

$$\left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

General Procedure: if $f(x)$ a fcn, $f^{-1}(x)$ its inverse fcn.

$$f(f^{-1}(x)) = x$$

$$\frac{d}{dx}f(f^{-1}(x)) = 1$$

$$f'(f^{-1}(x))\cdot\frac{d}{dx}f^{-1}(x) = 1$$

$$\boxed{\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

Practice: use $(\sqrt[3]{x})^3 = x$ to find $\frac{d}{dx}\sqrt[3]{x}$

$$\frac{d}{dx}(\sqrt[3]{x})^3 = \frac{d}{dx}x = 1$$

$$\cdot(\sqrt[3]{x})^2\cdot\frac{d}{dx}\sqrt[3]{x} = 1$$

$$3\left(\sqrt[3]{x}\right)^2 \cdot \frac{d}{dx}\sqrt[3]{x} = 1$$

$$\frac{d}{dx}\sqrt[3]{x} = \frac{1}{3\left(\sqrt[3]{x}\right)^2} = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}$$