Lecture 3: Limits, squeeze and continuity

Tuesday, January 17, 2017 12:36 PM

Practice Problems

1. $\lim_{x \to 1} \frac{x^2+2x-3}{x+2}$

2. $\lim_{x \to 1} \frac{x^2+2x-3}{x-1}$

3. $\lim_{x \to 1} \frac{\sqrt{x+8}-3}{x-1}$

BAD

ex 1. $\lim_{x \to 1} \frac{x^2+2x-3}{x+2} = \cancel{\frac{1^2+2-3}{1+2} = \frac{0}{3} = 0}$ (use limit laws)

$$= \frac{\lim_{x \to 1} x^2+2x-3}{\lim_{x \to 1} x+2}$$

(if both limits exist & bottom $\neq 0$)

$$= \frac{\lim_{x \to 1} x^2 + \lim 2x - \lim 3}{\lim_{x \to 1} x + \lim_{x \to 1} 2} = \frac{\left(\lim_{x \to 1} x\right)^2 + 2\lim_{x \to 1} x - \lim 3}{\lim_{x \to 1} x + \lim_{x \to 1} 2}$$

$$= \frac{(1)^2+2(1)-3}{1+2} = \frac{3-3}{3} = 0.$$

2. to simplify before limit:

$$\frac{x^2+2x-3}{x-1} = \frac{(x+3)(x-1)}{x-1} = x+3$$

if $x \neq 1$

3. Simplify

$$\frac{\sqrt{x+8}-3}{x-1} = \left(\frac{\sqrt{x+8}-3}{x-1}\right)\left(\frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}\right)$$

$$= \frac{(x+8)-9}{(x-1)(\sqrt{x+8}+3)} = \frac{x-1}{(x-1)(\sqrt{x+8}+3)}$$

$$= \frac{1}{\sqrt{x+8}+3}$$

$x \neq 1$

<u>Def</u> A function $f(x)$ is continuous at $x=a$ means that

$$\lim_{x \to a} f(x) = f(a)$$

<u>Extrapolating from examples</u>: $\boxed{\text{Polynomials are continuous!}}$

$$\lim_{x \to 5} x^3 - 2x + 1 = \cdots\cdots = \left(\lim_{x \to 5} x\right)^3 - 2\left(\lim_{x \to 5} x\right) + 1$$

$$= 5^3 - 2(5) + 1 \checkmark$$

Rational functions:

$$\lim_{x\to 3} \frac{x^2-2x+1}{3x^7-4x+5} = \frac{\lim_{x\to 3} x^2-2x+1}{\lim_{x\to 3} 3x^7-4x+5}$$

if bottom $\neq 0$.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)} \quad \text{if } g(a) \neq 0.$$

f, g polynomials

Conclusion: Rational functions are continuous wherever their denominators are $\neq 0$.

Best case scenario: (rational functions)

- Check if den $= 0$ when plug in

no → plug in

yes → simplify

Some other continuous functions:

Because of limit laws, if $f(x)$ & $g(x)$ are continuous at $x = a$ then so are:

1. $f(x) + g(x)$
2. $f(x)\,g(x)$

? $c\,f(x)$

3. $f(x)$ … if $g(a) \neq 0$

3. $c\,f(x)$

3. $\dfrac{f(x)}{g(x)}$ if $g(a) \neq 0$

<u>why?</u>.. explanation of 2.

if $\lim\limits_{x \to a} f(x) = f(a)$ (f cont) & $\lim\limits_{x \to a} g(x) = g(a)$

then limit law says

$$\lim_{x \to a} f(x)\,g(x) = f(a)\,g(a)$$

$f(x) = \sqrt{x}$ is continuous when defined ($x \geq 0$) $\quad \lim\limits_{x \to a} \sqrt{x} = \sqrt{a}$

<u>Another useful rule:</u>

if $f(x)$ is cont. at $x = a$ & $g(x)$ is cont at $x = f(a)$

then $g \circ f\,(x)$ is cont at $x = a$

$$a \xrightarrow{\;f\;} f(a) \xrightarrow{\;g\;} g(f(a)) = \lim_{x \to a} g(f(x))$$

$$f(a) = \lim_{x \to a} f(x) \qquad g(f(a)) = \lim_{x \to f(a)} g(x)$$

<u>ex:</u> $\dfrac{x-1}{x+3}$ cont at $x = 2$

$\rightsquigarrow \dfrac{2-1}{2+3} = \dfrac{1}{5}$ $\qquad \sqrt{x}$ cont at $x = \dfrac{1}{5}$

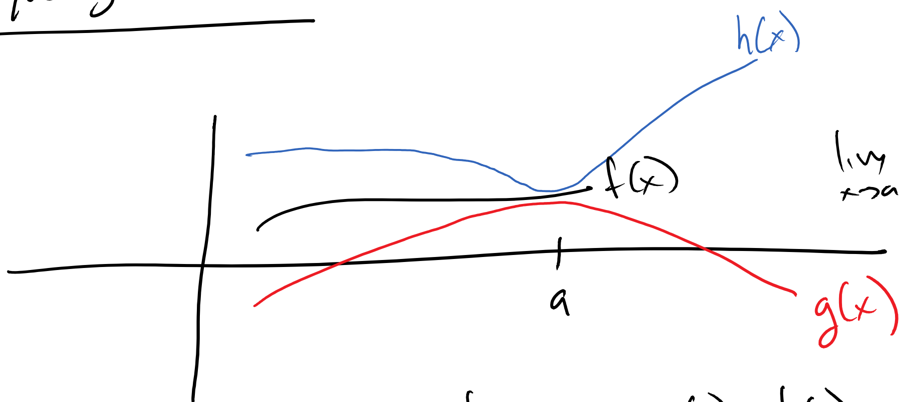
$$\sqrt{\frac{x-1}{x+3}} \quad \text{cont at } x=2$$

Same cont. functions (when defined)

$$\frac{\sqrt{x+3}-2}{\sqrt{x-1}-\sqrt[3]{\frac{x+3}{x+1/5x}}}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \qquad \lim_{x\to 0} \frac{e^x-1}{x}$$

Squeeze theorem



$\lim_{x\to a} f(x)$

if we have $g(x) \le f(x) \le h(x)$ (near a)

and limits: $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$

then $\lim_{x\to a} f(x) = L$ also.

How to find $g(x)$ & $h(x)$? no procedure.

warm up: $\lim\limits_{x\to 1} (1-x)^2 \sin^2\left(\frac{e^x}{1-x} - \sin x\right)$

$$-1 \le \sin(\quad) \le 1$$

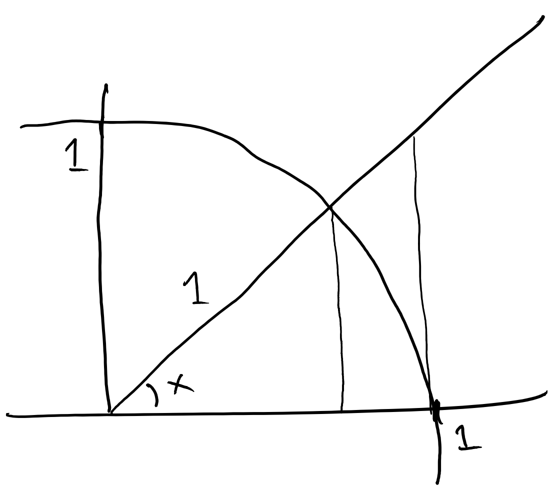
$$0 \le \sin^2(\quad) \le 1$$ mult. both sides by $(1-x)^2$

$$0 \le (1-x)^2 \sin^2(\quad) \le (1-x)^2$$

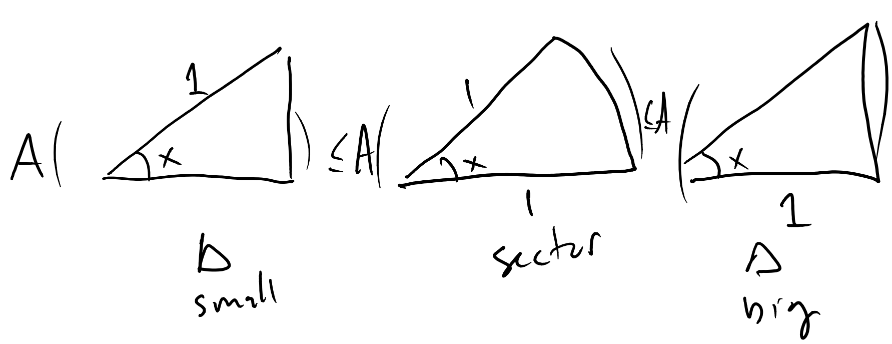
$\lim\limits_{x\to 1} 0 = 0$ $\qquad$ $\lim\limits_{x\to 1} (1-x)^2 = 0$

Squeeze $\Rightarrow \lim\limits_{x\to 1} (1-x)^2 \sin(\quad) = 0$

$$\lim_{x\to 0} \frac{\sin x}{x}$$

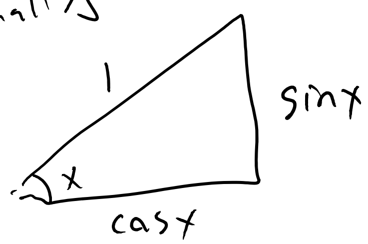


3 regions



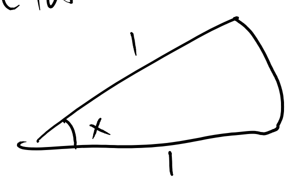
small Δ

small Δ



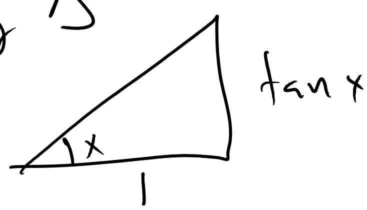
Area: $\frac{1}{2}\cos x \sin x$

sector:



$\frac{1}{2} r^2 \cancel{\theta} = \frac{1}{2} x$ $\quad$ $\measuredangle$ in radians.

big Δ



$A = \frac{1}{2}\tan x$

$$\frac{1}{2}\cos x \sin x \le \frac{x}{2} \le \frac{\tan x}{2}$$

$$\cos x \sin x \le x \le \tan x = \frac{\sin x}{\cos x}$$

$$\cos x \le \frac{x}{\sin x} \le \frac{1}{\cos x}$$

assume $\sin x > 0$ ($x > 0$ small)

$$\frac{1}{\cos x} \geq \frac{\sin x}{x} \geq \cos x$$

$$\lim_{x \to 0} \cos x = \cos 0 \quad (\text{cosine \& sine are continuous})$$

$$\phantom{\lim_{x \to 0} \cos x = } \text{"} 1$$

$$\lim_{x\to 0} \cos x = 1 = \lim_{x\to 0} \frac{1}{\cos x}$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = 1$$