Lecture 18: partial fractions, for real.

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Partial fraction practice:

1. $\dfrac{5x-7}{x^2-3x+2}$ 2. $\dfrac{1}{x^2-1}$

$x^2-3x+2=(x-2)(x-1)$

$$\frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2}+\frac{B}{x-1}$$

either → get rid of denominators

or → get common den on right, compare numerators

mult. everything by $(x-2)(x-1)$

$$5x-7 = (x-1)A+(x-2)B = Ax-A+Bx-2B$$

$$5x-7=(A+B)x+(-A-2B)$$

$\Longrightarrow$ $5=A+B$ $-7=-A-2B$

$7=A+2B$

$$\begin{array}{r} 7=A+2B \\ 5=A+B \\ \hline 2=B \end{array}$$

subtract

$A+B=5$

$A+2=5$

$A=3$

$$\frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} = \frac{3}{x-2} + \frac{2}{x-1}$$

$5 = A + B$ $\qquad$ $7 = A + 2B$

$5 - B = A$ $\qquad$ $7 = (5-B) + 2B$

$7 = 5 + B$

$2 = B$ $\qquad$ $5 = A + 2$

$A = 3$

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$= \frac{A}{x+1}\frac{(x-1)}{(x-1)} + \frac{B}{x-1}\frac{(x+1)}{(x+1)}$$

$$\frac{1}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

compare numerators $\rightsquigarrow$ $\quad 1 = A(x-1) + B(x+1)$

$$1 = Ax - A + Bx + B$$

$$0x + 1 = (A+B)x + (B-A)$$

$A + B = 0$ $\qquad$ $B - A = 1$

$$\searrow^{\text{add}}\swarrow$$

$2B = 1 \qquad B = \frac{1}{2}$

$A = -\frac{1}{2}$

$$\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{\frac{1}{2}}{x-1} - \frac{\frac{1}{2}}{x+1}$$

$$= \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

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3. $$\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x)$$

$$1 = Ax + 2A + Bx$$

$$0x + 1 = (A+B)x + 2A$$

$A + B = 0$

$2A = 1$

$B = -\frac{1}{2}$

$A = \frac{1}{2}$

$$\frac{1}{x(x+2)} = \frac{(\frac{1}{2})}{x} + \frac{(-\frac{1}{2})}{x+2}$$

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

$x = \sec u$?

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

$$\frac{1}{x^2-1} = \frac{1}{2}\,\frac{1}{x-1} - \frac{1}{2}\,\frac{1}{x+1}$$

$$\int \frac{1}{x+1}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|x+1| + C$$

$u = x+1$

$du = dx$

$$\int \frac{1}{2x+3}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$$

$u = 2x+3$

$du = 2\,dx$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|2x+3| + C$$

$$\int \frac{4}{(x+1)^2}\,dx = \int \frac{4}{u^2}\,du = \int 4u^{-2}\,du$$

$$= -4u^{-1} + C$$

$u = x+1$

$du = dx$

$$= -4(x+1)^{-1} + C.$$

$$\int \frac{x+4}{(\ldots)^2}\,dx \longrightarrow \text{partial fractions}$$

$$\int \frac{x+4}{(x+1)^2}\,dx \longrightarrow 1$$

$\xrightarrow{u}$

parts

$u = x+4$

$dv = \dfrac{dx}{(x+1)^2}$

$u = x+1$

$du = dx$

$$\int \frac{u+3}{u^2}\,du$$

$$= \int (u+3)u^{-2}\,du$$

$$= \int (u^{-1}+3u^{-2})\,du$$

$$\frac{x+4}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{x+1}$$

solve by mult. to clear denominators

or setting common den.

$$\frac{x+4}{(x+1)^2} = \frac{A}{(x+1)^2} + \frac{B(x+1)}{(x+1)^2} = \frac{A+Bx+B}{(x+1)^2}$$

$$x+4 = Bx + (A+B)$$

$B = 1$

$A+B = 4$

$A = 3$

$$\frac{x+4}{(x+1)^2} = \frac{3}{(x+1)^2} + \frac{1}{x+1}$$

$$\int \frac{x+4}{(x+1)^2}\,dx = \int \frac{3}{(x+1)^2}\,dx + \int \frac{1}{x+1}\,dx$$

$$= -\frac{3}{x+1} + \ln|x+1| + C$$

General Picture:

$\deg f < 3$

Given $$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

cf.

$$\frac{3x^3-2x+7}{(x-1)(x+3)(x+7)(x-2)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x+7} + \frac{D}{x-2}$$

repeated roots

$$\frac{3x^2-2x+1}{(x-1)^2(x+3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3}$$

$$\frac{x+1}{(x+3)^4(x-1)} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x+3)^4} + \frac{E}{x-1}$$

$$\int \frac{3}{x^2+1}\,dx \qquad \int \frac{3x+4}{(x^2+1)(x-2)}\,dx$$

$\tan u = x$

$$\frac{3x+4}{(x^2+1)(x-2)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-2}$$

$$3x+4 = (Ax+B)(x-2) + C(x^2+1)$$

$\vdots$

$$\int \frac{Ax+B}{x^2+1}\,dx + \int \frac{C}{x-2}\,dx$$

$$\int \frac{7x-1}{x^2+1}\,dx = \int \frac{7x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx$$

$u = x^2+1$

trig

$\tan u = x$

$$\int \frac{x^{19}-3x^3+2x-7}{(x^2+1)^2(x+3)^2}\,dx$$