Lecture 30: computing the radius of convergence, and life on the boundary

Wednesday, November 5, 2014 12:24 PM

Question: Given a power series

$$\sum_{n=0}^{\infty} b_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} b_n (x-c)^n$$

for which values of x does series converge?

ex: "$f(x)$" $= \sum_{n=1}^{\infty} \underbrace{\frac{(3x-2)^n}{n}}_{a_n}$ (similar, but slightly different form)

ratio test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{(3x-2)^{n+1}}{n+1}\right)}{\left(\frac{(3x-2)^n}{n}\right)}$

$$= \lim_{n\to\infty} \frac{(3x-2)^{n+1}}{(3x-2)^n} \cdot \left(\frac{n}{n+1}\right) = \lim_{n\to\infty} (3x-2)\left(\frac{n}{n+1}\right)$$

$\to 1$

$= 3x-2 =$ "ρ"

Conclusion will be:

$|3x-2| < 1$ converges

$|3x-2| > 1$ diverges

$|3x-2| = 1$?

$|3x-2|<1$

$-1<3x-2<1$

$1<3x<3$

$\frac{1}{3}<x<1$ ← converges

$x>1$ or $x<\frac{1}{3}$
diverges

$x=1$ or $\frac{1}{3}$?

Absolute/Conditional convergence

$\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum |a_n|$ converges.

Fact: If $\sum a_n$ converges absolutely then it converges.

Converse not necessarily true: possible to converge, but not absolutely.

We say $\sum a_n$ converges conditionally if it converges but $\sum |a_n|$ diverges.

Theorem "Alternating series test"

Given a series $u_1 - u_2 + u_3 - u_4 + u_5 - \cdots$

$\sum_{1}^{\infty} (-1)^{n+1} u_n$ with each $u_i > 0$

$n=1$

then series will converge if

- $\lim_{n\to\infty} u_n = 0$
- $u_{n+1} \le u_n$

ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

conditionally converges.

Intervals of convergence for power series

"Radius"

Given a power series: $\sum a_n (x-c)^n$

either • there is some $r>0$ such that
if $|x-c|<r$ then series converges absolutely
and $|x-c|>r$ then diverges

or • series converges abs. for all x
("$r=\infty$")

or • series diverges for all $x \neq c$
("$r=0$")

$\sum^{\infty} 2^n \ldots x^n$

Ex: $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{2^n}{n}(x-1)^n}_{a_n}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{2^{n+1}}{n+1}(x-1)^{n+1}\right|}{\left|2^n/n\,(x-1)^n\right|} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n}\left(\frac{n}{n+1}\right)|x-1|$$

(testing for absolute convergence)

$$= \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)|x-1|$$

$$= 2|x-1| = \text{"}\rho\text{"}$$

so converges if $2|x-1| < 1$

$$|x-1| < \tfrac{1}{2}$$

$$\text{"}|x-c| < r\text{"}$$

diverges if $2|x-1| > 1$

know converge for $|x-1| < 1/2$ diverge $|x-1| > \frac{1}{2}$

$$-\tfrac{1}{2} < x-1 < \tfrac{1}{2} \qquad\qquad x > \tfrac{3}{2}$$

$$\tfrac{1}{2} < x < \tfrac{3}{2} \qquad\qquad x < -\tfrac{1}{2}$$

what about $x = \frac{1}{2}, \frac{3}{2}$?

$$x = \tfrac{1}{2} \to \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\tfrac{1}{2}-1\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(-\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges by alternating series

$$\sum^{\infty} 2^n\left(\tfrac{3}{2} \ldots\right)^n = \sum^{\infty} 2^n \cdot \left(\tfrac{1}{2}\right)^n = \sum \tfrac{1}{}$$

$x = \frac{3}{2}$ $\quad \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{3}{2}-1\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\cdot\left(\frac{1}{2}\right)^n = \sum \frac{1}{n}$

diverges (harmonic)

converge exactly for x in $\left[\frac{1}{2}, \frac{3}{2}\right)$

diverges everywhere else!