Preliminaries

Rings: always associative & unital. not necessarily comm.

homomorphisms always unital.

$0=1$ is allowed.

Modules: $_RM$ "M a left R-mod"

left modules, right modules N_S "N right S-mod"

$_RP_S = P$ R-S bimod.

Given rings R, S an R-S bimodule M is an ab. gp w/ both a left R-mod & right S-mod structures s.t. $r(ms) = (rm)s$ all $r \in R, s \in S, m \in M$.

Structure theory

Def Let R be a ring a left R mod P is simple if it has no proper nonzero submodules

Def R ring P a left R-mod, $X \subset P$ a subset, then

$$\mathrm{ann}_R(X) = \{ r \in R \mid rx = 0 \text{ all } x \in X \}$$

Note: this is always a left ideal, & is a 2-sided if $X = P$.

$r \in \mathrm{ann}_R(X)$ $rsx = 0$?

$r(sx)$
$sx \in X$? ok if $X = P$.

Def $I \lhd R$ = ideal
$I \lhd_\ell R$ = l. ideal
$I \lhd_r R$ = r. ideal.

Def $I \lhd R$ is left primitive if it is of the form
$I = \mathrm{ann}_R(P)$ P simple.

Prop: Suppose P is a $\neq 0$ right R-module, Then TFAE

1. P simple
2. $mR = P$ all $m \in P \setminus \{0\}$
3. $P = R/I$ some $I \lhd_r R$ max'l.

Def a left R-mod P is semisimple if $P \simeq \bigoplus_{i=1}^{n} P_i$
each P_i simple.

Aside
if F is a field, then an F-algebra is a ring A together w/ an F-vector space structure st. $\forall \lambda \in F$ $a, b \in A$,
$$(\lambda a)b = \lambda(ab) = a(\lambda b)$$
$F \hookrightarrow A$ $\quad F \hookrightarrow Z(A)$
$\lambda \longmapsto \lambda \cdot 1$

Prop let A be an algebra over a field F

M a semisimple left A-module, f.dim'l as an F-vectorspace, $P < M$ a submodule. Then P & M/P are semisimple, and can find $P^\perp < M$ s.t. $M \simeq P \oplus P^\perp$.

Pf sketch:

to show M/P semisimple

choose $Q < M/P$ max'l ssimple. suppose $Q \neq M/P$

then can find M_i summand of M w/ $\overline{M_i} \not\subseteq Q$

since $\overline{M_i}$ image of M_i simple & $\overline{M_i} \neq 0 \Rightarrow \overline{M_i} \simeq M_i$

$\overline{M_i} \cap Q \subsetneq \overline{M_i}$ simple $\Rightarrow \overline{M_i} \cap Q = 0$

$\Rightarrow Q \oplus \overline{M_i}$ ssimple, larger ↯.

Def R ring, $J_r(R) = \bigcap$ all max'l r. ideals

$J_\ell(R) = \bigcap$ all max'l l. ideals

will show later $J_\ell = J_r$.

Note: {annihilators of elmts in simple r. mods} = {max'l r. ideals}

$\Rightarrow J_r(R) = \bigcap$ all annihilators of simple r. mods

$= \bigcap_{\substack{M \text{ simple} \\ \text{r. mod's}}} \operatorname{ann}_R(M) \Rightarrow J_r(R) \lhd R$!

Lem Suppose A f.dim'l F-algebra. Then A_A is a semisimple right A module $\iff J_r(A)=0$

Pf: if A_A s.simple $\Rightarrow$ $A_A = \bigoplus P_i$ simple $\Rightarrow$

$\bigoplus_{j\neq i} P_j \triangleleft_r A$ max'l r. ideal. $\Rightarrow \bigcap$ these get 0

so $J_r(A)=0$.

if $J_r(A)=0$ then $\exists$ finite collection of max'l r. ideals I_i

s.t. $\bigcap I_i = 0$ $\Rightarrow$ $A \longrightarrow \bigoplus A/I_i$ = s.simple
injective

$\rightarrow A$ semisimple. "D".

Interlude:

$J_r = J_l$?

Recall $r\in R$ is left invertible if $\exists s\in R$ s.t. $sr=1$
right " " $\exists s\in R$ s.t. $rs=1$

Can have left but not right invertibility:

$\mathrm{End}\left(\bigoplus_{i=0}^{\infty} F\right)$

$(\lambda_0, \ldots, \lambda_n, \ldots)$

$\downarrow$ left invertible, not right.

$(0, \lambda_0, \ldots -)$

<u>Aside</u>: If A is f.d. alg./F, $a \in A$ right invertible $\Leftrightarrow$ left-invertible

<u>Pf</u>: pick $a \in A$ $\quad A \xrightarrow{T} A$ $\quad$ lin. trans. of F-spaces.

$x \longmapsto ax$

a right inv $\Rightarrow$ surjective

$ab = 1 \Rightarrow \forall y \in A, \quad a(by) = y \Rightarrow T$ bijective

$\det T \neq 0$

$\chi_T(t) = t^n + c_{n-1} t^{n-1} + \cdots + c_0 \qquad c_0 = \pm \det T$

Cayley-Hamilton thm $\Rightarrow \chi_T(T) = 0$

$$(-c_0^{-1})\underbrace{(a^{n-1} + c_{n-1}a^{n-2} + \cdots + c_1)a}_{-c_0 a} = 1$$

$\Rightarrow ((-c_0^{-1})(a^{n-1} + \cdots c_1)$ is a left inv. to a

$\square$

<u>Lemma</u> R ring $r \in R$ $s, t \in R$ s.t. $sr = 1 = rt \Rightarrow s = t$.

Pf: $s = s \cdot 1 = srt = 1t = t$

Def R a ring, $r \in R$. r is l. quasiregular if $1-r$ is l. invertible.

sim. right quasiregular

r is quasi-regular if it is both left & right q. regular.

Lem suppose $I \trianglelefteq_r R$ s.t. all elements of I are r. quasi-regular

$\Rightarrow$ all elements of I are quasiregular.

Proof: let $x \in I$ WTS $(1-x)$ has a left inverse.

know $\exists s$. $(1-x)s = 1$. Let $y = 1-s$ $s = 1-y$

$(1-x)(1-y) = 1$ $\Rightarrow xy - x - y = 0$

$1 - x - y + xy$ $y = xy - x$ $x \in I$

$xy \in$

$\Rightarrow y \in I$.

$\Rightarrow y$ r. quasi regular $(1-y)$ r. invertible.

but $(1-y)$ is also left. invertible

w/ inverse $(1-x)$

$\Rightarrow (1-x)(1-y) = 1 \Rightarrow (1-y)(1-x) = 1$ $\Rightarrow 1-x$ is left. inv.

$\Rightarrow$ left q. reg.

$\square$.

Lem let $x \in J_r(R)$. Then x is q. regular.

(resp. $J_e(R)$)

Pf: suffices to show, $\forall x \in J_r(R)$ x is r. q-regular.

$x \in J_r(R) \Rightarrow x \in$ all max'l r. ideals $\Rightarrow 1-x \in$ no max r. ideals

$\Rightarrow (1-x)R = R \Rightarrow (1-x)s = 1$ some s. $\square$.

Lem Suppose $I \triangleleft R$ s.t. all elmts are q-reg. Then $I \subset J_r(R)$ (& $J_\ell(R)$)

Pf: Suppose $K < R$ max'l right.

WTS: $K \supset I$.

Consider $K + I$. if $I \not\subset K$ then $K + I = R$

$\Rightarrow 1 = k + x \quad k \in K, x \in I \Rightarrow k = 1 - x \Rightarrow k$ invertible ~~↯~~

$\Rightarrow I \subset K$ as claimed $\square$.

Cor $J_r(R)$ = unique ideal max'l w/ respect to the property that each of its elmts is q-regular.

$J_r(R) = J_\ell(R)$

$\Rightarrow J(R) = J_r(R) = J_\ell(R)$.

Def: R is called semiprimitive if $J(R) = 0$.

Thm (Schur's lemma) Let P be a simple right R-module $D = \mathrm{End}_R(P_R)$. Then D is a division ring.

Remark: D acts on P on left & P has a natural D-R bimodule structure.

Pf: suppose $f \in D \setminus \{0\}$

$\ker f, \operatorname{im} f < P$ as right R-modules.

$\Rightarrow \ker f \neq P \Rightarrow \ker f = 0$ (simple) $\Rightarrow \operatorname{im} f \neq 0$

$\Rightarrow \operatorname{im} f = P$

$\Rightarrow f$ is bijective.

set f^{-1} = the inverse, can check f^{-1} is right R-linear

$\Rightarrow f^{-1} \in D$. $\square$

Approach $\quad R \hookrightarrow \operatorname{End}_R(R_R) = R$

$R_R = \bigoplus P_i$

Endomorphisms of semisimple modules

$M = \bigoplus_{i=1}^{m} M_i \qquad N = \bigoplus_{j=1}^{n} N_j \qquad M_i, N_j$'s simple right R-modules.

If $f : M \longrightarrow N$ is an right R-mod hom.

$f_j = f|_{M_j} \qquad f_j$ is a tuple $(f_{1,j}, f_{2,j}, f_{3,j}, \dots f_{n,j})$

$f_{i,j} : M_j \longrightarrow N_i$

Can represent f as a matrix

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$$f = \begin{bmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,m} \\ \vdots & & & \vdots \\ f_{n,1} & \cdots & \cdots & f_{n,m} \end{bmatrix} \quad \text{acting on} \quad \begin{bmatrix} m_1 \\ \vdots \\ m_m \end{bmatrix}$$

i.e. $$\mathrm{Hom}_R(M_R, N_R) = \begin{pmatrix} \mathrm{Hom}_R(M_1,N_1) & \cdots & \mathrm{Hom}_R(M_1,N_m) \\ & & \vdots \\ & & \end{pmatrix}$$

w/ standard matrix mult. by composition

<u>Theorem (Wedderburn-Artin)</u>

Let A be finite dim'l over a field and $J(A)=0$.

Then we may write $A = \bigoplus_{i=1}^{n} P_i^{d_i}$, P_i mutally non isom.

and $A \simeq \prod_{i=1}^{n} M_{d_i}(D_i)$, $D_i = \mathrm{End}(P_i)$ a div. alg.

<u>Pf:</u> $A \simeq \mathrm{End}_A(A_A)$ $\quad J(A)=0 \Rightarrow A_A = \bigoplus P_i^{d_i}$

Schur $\Rightarrow D_i = \mathrm{End}_A((P_i)_A)$ is division

$[P,Q] = \mathrm{Hom}_A(P,Q)$

$$\mathrm{End}_A(A_A) = \begin{bmatrix} [P_1^{d_1}, P_1^{d_1}] & [P_1^{d_1}, P_2^{d_2}] & \cdots \\ \vdots & & \\ & & \end{bmatrix}$$

$$[P_i^{d_i}, P_j^{d_j}] = \begin{array}{c} \left.\begin{bmatrix} [P_i, P_j] & [P_i, P_j] & \cdots \\ \vdots & & \\ & & \end{bmatrix}\right\} d_j \\ \underbrace{\hphantom{[P_i,P_j] \quad [P_i,P_j] \quad \cdots}}_{d_i} \end{array}$$

$[P_i, P_j] = 0$ unless $i = j$

else: $D_i = \mathrm{End}(P_i)$ if $i = j$
division

$$\mathrm{End}_A(A_A) = \begin{bmatrix} M_{d_1}(D_1) & & & 0 \\ & M_{d_2}(D_2) & & \\ & & \ddots & \\ 0 & & & M_{d_n}(D_n) \end{bmatrix} = M_{d_1}(D) \times \cdots \times M_{d_n}(D_n)$$

$\square$.

<u>Cor</u> If A is a finite dim'l simple F-algebra, then

$A \simeq M_n(D)$ D a div. alg $/F$, and moreover
$Z(A) = Z(D)$

Pf: $J(A) \trianglelefteq A$ A simple $J(A)=0$.

$A = \overset{n}{\underset{i=1}{\times}} M_{d_i}(D_i)$. But each factor $M_{d_i}(D_i)$ is an ideal.

so there can only be 1 since A is simple.

note: if $a = \begin{bmatrix} d_{11} & d_{12} & \cdots \\ & \vdots & \\ & & \end{bmatrix} \in Z(A) \Rightarrow d_{ij} = 0$ if $i \neq j$

(commute w/ $\begin{bmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & \ddots 0 \end{bmatrix}$

but check if $a = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$ commutes w/ e_{ij}

$\Leftrightarrow d_i = d_j$

$\Rightarrow$ if $a \in Z(A) \Rightarrow a = \begin{bmatrix} d & & \\ & \ddots & \\ & & d \end{bmatrix} = d \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

if this commutes w/ all $\begin{bmatrix} d' & & \\ & \ddots & \\ & & d' \end{bmatrix} \Rightarrow d \in Z(D)$

conversely, if $d \in Z(D)$ then $\begin{bmatrix} d & \\ & d \end{bmatrix} = (id) \cdot d \in Z(A)$.

D.

Definition An F-algebra A is called a central simple algebra over F (CSA/F) if

A simple, $Z(A) = F$.