Lecture 11: Ramification

Friday, April 10, 2015 2:33 PM

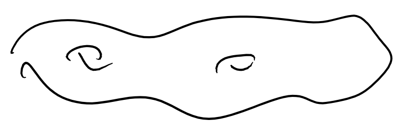
ramification "understandable"

the rest "somewhat mysterious"

Most deeply "come from" ramification

$\Updownarrow$

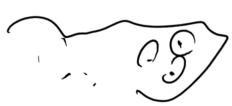
finiteness of Ш



H_1

poke it!

don't poke



ramified

unram

Recall the example

L/F cyclic Galois $\langle\sigma\rangle = \mathrm{Gal}(L/F)$ $[L:F]=n$

$$L[x;\sigma] = L \oplus Lx \oplus Lx^2 \oplus \cdots$$

$$x\ell = \sigma(\ell)x \qquad \ell x^i m x^j = \ell\sigma^i(m)x^{i+j}$$

$$Z(L[x;\sigma]) = F[x^n]$$

also can see: $L[x;\sigma]$ a domain free mod / $F[x^n]$ of rank n^2 (basis $\{l_i x^j\}$ l_i basis for L/F, $0 \le j \le n-1$)

Set $D = L[x;\sigma] \otimes_{F[x^n]} F(x^n)$ still a domain (domain $\Rightarrow$ torsion free $\Rightarrow$ localization can't kill anything)

n^2 dim'l vs / $F(x^n)$

domain $\Rightarrow$ div. alg.

"nicely semiramified"

how is D built? $t = x^n$

$K = F(t)$ $E = L(t)$

- field K
- elmt $t \in K$, prime in some ring $R \subset K$ w/ fraction field K (PID, ...)
- extension E/K, "independent from t" cyclic Galois, $\langle\sigma\rangle$

$(E/K, \sigma, t)$

<u>Discrete Valuation rings</u>

<u>Def/Lem</u> TFAE R ring

1. R is a disc. val. ring

2. R is a local PID

$2\frac{1}{2}$. R a 1dim'l reg. loc. Noeth. ring.

3. R is normal noeth, 1 dim'l, local domain

4. $\exists$ valuation

$$v : R \setminus \{0\} \longrightarrow \mathbb{Z}_{\geq 0} \quad \text{(WLOG surjective)}$$

$$\left.\begin{array}{l} v(rs) = v(r) + v(s) \\ v(r+s) \geq \min\{v(r), v(s)\} \end{array}\right\} \text{valuation}$$

$$\text{\{} \ v(r) = 0 \iff r \in R^*$$

ex: $\mathbb{Z}_p$ $\quad v(a) =$ how many times p divides a.

$$k[[t]] \quad v\left(\sum_{i=v}^{\infty} a_i t^i\right) = v$$

$$a_v \neq 0$$

R regular Noeth ring $\quad \mathfrak{p}$ ht 1 prime,

$R_{\mathfrak{p}}$ = dvr.

ex: R Noeth. UFD $\quad r \in R$ looks like

$$r = u \pi_1^{e_1} \pi_2^{e_2} \cdots$$

$$v_1(r) = e_1 \qquad v_1(\pi_2) = 0$$

$R_{(\pi_1)}$ is a dvr w/ valuation v_1.

If R is dvr, then v extends to frac(R)

$$v\left(\frac{a}{b}\right) = v(a) - v(b)$$

Furthermore: can check that if we define

$|x| = e^{-v(x)} \in \mathbb{R}$ then this defines a norm on R & F

$|x| = 0$ if $x = 0$.

& dist function $d(x,y) = |x-y|$

& R, F are metric spaces.

Important case: R is complete w/r/t d.

in this case, we say R is a complete dvr.

F a complete dvf

(yes! F is also complete in this case).

lots of $\mathbb{Q}$ above each — lots above 1 prime — 1 prime to some powr.

$$\mathbb{Z} \rightsquigarrow \mathbb{Z}_{(p)} \rightsquigarrow \mathbb{Z}_p$$

lots of primes — 1 prime — 1 prime

$$\begin{array}{ccl} L & \supset & S = \{x \in L \mid x \text{ integral over } R\} \\ | \text{sep} & & | \\ F & \supset & R = \text{PID (or Dedekind ...)} \end{array}$$

frac(R)

$P \lhd R$ prime $\quad PS = Q_1^{e_1} Q_2^{e_2} \cdots Q_g^{e_g} \quad Q_i$'s primes in S let F a cdvf. D/F div alg.

<u>fact</u>: valuation $v: F^* \to \mathbb{Z}$ can always be extended uniquely to D

$$w: D^* \longrightarrow \mathbb{Q} \qquad w|_F = v$$

$$w(d_1 d_2) = w(d_1) + w(d_2)$$

$$w(d_1 + d_2) \geq \min\{w(d_1), w(d_2)\}$$

$$B = \{d \in D \mid v(d) \geq 0\} \qquad R = \{x \in F \mid v(x) \geq 0\}$$

$$m_B = \{d \in D \mid v(d) > 0\} \qquad m_R = \{x \in F \mid v(x) > 0\}$$

can find: B/m_B is a div. alg. R/m_R a field in B/m_B

assuming perfect.

$$\mathrm{im}(w) = \frac{1}{e}\mathbb{Z} \subset \mathbb{Q}$$

<u>Draxl's version of Ostrowski's thm</u> $[B/m_B : R/m_R] \cdot e = [D:F]$

ram. index of D

$$\begin{array}{ccc} \overline{D}' \otimes_{R/m_R} Z & \simeq & B/m_B \\ | & & | \text{ c.d.a.} \\ \overline{D}' & & Z(B/m_B) \\ & & | e \\ & & R/m_R \end{array}$$

(unramified) ramification

$$D = D' \otimes (L/F, \sigma, t)$$

$$v(t) = 1$$

$L/F \longleftrightarrow Z(B/m_B)/R/m_R$

deg e

$$D' \hookrightarrow \overline{D}'$$