Lecture 12: Ramification, part 2

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(discrete) Valuation on a field

$v: F^* \longrightarrow \mathbb{Z}$

$v(ab) = v(a) + v(b)$

$v(a+b) \geq \min\{v(a), v(b)\}$

same for D a div alg

$v: D^* \longrightarrow \mathbb{Z}$

$V_D, \mathcal{O}_D = \{d \in D \mid v(d) \geq 0\}$

$m_D = \{d \in D \mid v(d) > 0\}$

$\overline{D} = \mathcal{O}_D / m_D$ res. division alg.

So, given $F \subset D$ D cda. $v: D \to \mathbb{Z}$ (restrict to F)

can consider
$$\begin{array}{ccc} \mathcal{O}_D & \supset & \mathcal{O}_F \\ \cup & & \cup \\ m_D & \supset & m_F \end{array} \qquad \overline{F} \subset \overline{D}$$

given $d \in D^*$, inn_d preserves $\mathcal{O}_D$ & m_D

inn_d induces $\overline{\mathrm{inn}_d}$ on $\overline{D}/\overline{F}$

$Z(\overline{D})$ may not be $\overline{F}$

$$\begin{array}{ll} \overline{D} & \\ \mid & \Big\} \text{ central} \\ Z(\overline{D}) & \\ \mid & \Big\} \text{ see w/ ramification} \end{array}$$

$$\begin{array}{c} Z(\bar{D}) \\ | \\ \bar{F} \end{array} \Big] \text{ see w/ ramification}$$

Let's assume that F is complete w/r/t $v|_F$

Facts:

- If (F,v) is a c.d.val. field and L/F finite, then $\exists!$ extension of v to L $\quad (L^* \xrightarrow{v} \mathbb{Q})$ (Serre Loc. Fields)
- If (F,v) is a cdvf, D/F division, then $\exists!$ ext. of v to D. (Morandi)

$$\Gamma_D = v(D^*)$$

$$\Gamma_F$$

$$\begin{tikzcd} D^* \arrow[r] \arrow[d, "v"] & \mathrm{Aut}\left(Z(\bar{D})/\bar{F}\right) \\ \Gamma_D/\Gamma_F \arrow[ur] & \end{tikzcd}$$

$d \in \mathcal{O}_D^*$

$\bar{d} \in \bar{D}$

$d \in \mathcal{O}_D^* \quad x \in \bar{D}$

$\tilde{x} \in \mathcal{O}_D$

$\overline{\mathrm{inn}_d}(x) = \mathrm{inn}_{\bar{d}}(x)$

$x \in \bar{D}$

$\overline{d\tilde{x}d^{-1}} = \mathrm{inn}_{\bar{d}}(x) \quad \rightsquigarrow$ if $x \in Z(\bar{D}) \Rightarrow \overline{\mathrm{inn}_d}(x) = x$ for $d \in \mathcal{O}_D^*$

we just showed $D^* \longrightarrow \mathrm{Aut}\left(Z(\bar{D})/\bar{F}\right)$

factors through $D^*/\mathcal{O}_D^* \cong \Gamma_D \qquad \mathcal{O}_D^* \longrightarrow D^* \xrightarrow{v} \mathbb{Z}$

<u>Fact:</u> (Jacob & Wadsworth)

get an $\cong$ $\quad \Gamma_D / \Gamma_F \xrightarrow{\sim} \mathrm{Aut}(Z(\overline{D}) / \overline{F})$

and
$$\begin{array}{c} Z(\overline{D}) \\ \Big| \text{ insep.} \\ L \\ \Big| \text{ cyclic Galois} \\ \overline{F} \end{array}$$

<u>Def</u> D/F (F cdvf) is <u>tame</u> if $Z(\overline{D}) / \overline{F}$ is separable.

"inertially split"

<u>Assume</u> tame!

$$\begin{array}{c} \overline{D} \\ \Big| \\ Z(\overline{D}) \\ \Big| \\ \overline{F} \end{array} \qquad \overline{D} \simeq \overline{D}_0 \otimes_{\overline{F}} Z(\overline{D})$$

some $\overline{D}_0 / \overline{F}$ CSA

<u>Unramified extensions</u>

E/F $\qquad \Gamma_E \supset \Gamma_F$ $\quad$ if $=$, we say E/F is unramified

$|\Gamma_E / \Gamma_F|$ is ram. index.

$\overline{E} / \overline{F}$ res. ext.

$[\overline{E} : \overline{F}]\,[\Gamma_E : \Gamma_F] = [E : F]$

Ostrowski

... extns (Arith. hom. lifting property) if F cdvf $\bar{F}$ residue then there is an equiv. of cats:

$$\left\{ \begin{matrix} \bar{L} \\ | \\ \bar{F} \end{matrix} \ \text{sep} \right\} \longleftrightarrow \left\{ \begin{matrix} L \\ | \\ F \end{matrix} \ \text{unram} \right\}$$

(Serre)

also an equiv of cat

$$\left\{ \begin{matrix} \bar{D} \\ | \\ \bar{F} \end{matrix} \ \begin{matrix} \text{central} \\ \text{division} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} D \\ | \\ F \end{matrix} \ \begin{matrix} \text{c.div.} \\ \text{unramified} \end{matrix} \right\}$$

$\Gamma_D = \Gamma_F$

(Jacob-Wadsworth)

$D =$ tame $/F$

$(\dim_{\bar{F}} \bar{D}) [\Gamma_D : \Gamma_F] = \dim_F D$

$(\deg \bar{D})^2 [Z(\bar{D}) : \bar{F}] [\Gamma_D : \Gamma_F] = (\deg_F D)^2$

$\Gamma_D / \Gamma_F \simeq \mathrm{Gal}(Z(\bar{D}) / \bar{F}) \Rightarrow [\Gamma_D : \Gamma_F] = [Z(\bar{D}) : \bar{F}]$

$\Rightarrow (\deg \bar{D})(\mathrm{ram}\, D) = \deg D$

$D \cong D_0 \otimes (\qquad)$

ramification information

$$\begin{matrix} \bar{D} \\ | \\ Z(\bar{D}) \end{matrix} \qquad \bar{D}_0$$

choose $\pi \in D^*$, $v(\pi)$ generating Γ_D positive

$\overline{\mathrm{inn}_{-}} \in \mathrm{Aut}(\bar{D})$ extends $\overline{\mathrm{inn}_\pi} \in \mathrm{Aut}(Z(\bar{D})/\bar{F})$

$$\begin{array}{ccc} & & \overline{D}_0 \\ Z(\overline{D}) & & \\ | & / & \\ \overline{F} & & \end{array}$$

$\overline{\mathrm{inn}_\pi} \in \mathrm{Aut}(\overline{D})$ extends $\overline{\mathrm{inn}}_\pi \in \mathrm{Aut}(Z(\overline{D})/\overline{F})$

$$(\overline{\mathrm{inn}_\pi})^{\overset{[\Gamma_D:\Gamma_F]}{\underset{e}{\shortparallel}}} = (\overline{\mathrm{inn}_{\pi^e}})$$

$v(\pi^e) \in \Gamma_F$

<u>Claim</u>: can alter choice of π (via $Z(\overline{D})$) so that π^e induces trivial action on $\overline{D}$.

(use Hilbert 90)

<u>Afterwards</u>: descent $\overline{D}_0 = (\overline{D})^{\overline{\mathrm{inn}_\pi}}$, then this satisfies $\overline{D}_0 \otimes_{\overline{F}} Z(\overline{D}) = \overline{D}$

$$\text{Lift} \quad \begin{array}{c} Z(\overline{D}) \\ | \\ \overline{F} \end{array} \quad \text{to} \quad \begin{array}{c} L \\ |\ \text{unram} \\ F \end{array} \subset \begin{array}{c} D \\ | \\ F \end{array}$$

<u>Claim</u>: subalg of D gen. by (a good choice of) $\pi \in D$ "val 1" and L, then π induces Gal action on L/F via conjugation

$$(L/F, \underset{\mathrm{inn}\,\pi}{\sigma}, \underset{t\, =\, \text{"val 1" in } F}{\pi^e})$$

<u>Structure thm (tame case)</u>:

$$D \cong D_0 \otimes (L/F, \sigma, t)$$

lift of $\overline{D}_0$ "nicely semiramified"

<u>Def</u> D is tame if $\exists\, L/F$ unramified w/ $D \otimes_F L$ split.

$$\begin{array}{ccc} F^{sep} & & \\ \Big| G & & \bar{L} \\ F & & G(\bar{L}/\bar{F}) \end{array} \qquad \begin{array}{ccc} F^{unr} & & \\ \Big| G & & L \\ F & & G(\bar{L}/\bar{F}) \end{array}$$

$$D \longleftrightarrow H^2(G, F^{unr\,*}) \overset{?}{\longleftrightarrow} H^2(G, \mathcal{O}_{F^{unr}}^*)$$
$$\overset{\shortparallel}{Br}(F^{unr}/F)$$

$$0 \to \mathcal{O}_L^* \to L^* \xrightarrow{v} \mathbb{Z} \to 0$$

$G(L/F)$ acts on everything.

$$H^2(G, \mathcal{O}_L^*) \to H^2(G, L^*) \to H^2(G, \mathbb{Z})$$

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

$$\underset{\overset{\shortparallel}{0}}{H^1(G,\mathbb{Q})} \to H^1(G, \mathbb{Q}/\mathbb{Z}) \to H^2(G,\mathbb{Z}) \to \underset{\overset{\shortparallel}{0}}{H^2(G,\mathbb{Q})}$$

order of elements in $H^i(G, -)$ finite is finite.

$|G|$-torsion & divisible.

$\text{Hom}(G, \mathbb{Q}/\mathbb{Z})$

$$H^2(G, \mathcal{O}_L^\times) \to H^2(G, L^\times) \xrightarrow{\text{ram.}} \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

$$[D] \rightsquigarrow \left(Z(\bar{D})/F,\ \overline{\mathrm{inn}_\pi}\right)$$

exercise: subgps of $\mathbb{Q}/\mathbb{Z}$ are all cyclic of the form $\mathbb{Z}\cdot 1/m$

so an elmt of $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \ni \chi$ $\quad \ker\chi \triangleleft G$

moreover, the identification

$$\begin{tikzcd}[column sep=small]
 & L \arrow[dl, dash, "\ker\chi"'] & \\
K \arrow[dr, dash, "G/\ker\chi"'] & & \\
 & F &
\end{tikzcd}$$

$$G/\ker\chi = \mathrm{Gal}(K/F) \simeq \tfrac{1}{m}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$
$$\sigma \leftarrow\!\!\rightsquigarrow \tfrac{1}{m}$$

cyclic $\to$ $G/\ker\chi$

gives $\chi \longleftrightarrow (K/F, \sigma)$ $\quad \langle\sigma\rangle = \mathrm{Gal}(K/F)$

Observation: if $D \simeq D_0 \otimes (K/F, \sigma, t)$

then $\mathrm{ram}[D] = (K/F, \sigma)$

Pf: WLOG can assume $D = D_0$ or $D = (K/F, \sigma, t)$

in case $D = D_0$, since D_0 is a lift of $\overline{D_0}$

it turns out that this means that we can lift cocycle for $\overline{D_0}$

in $H^2(G, \bar{L}^*)$ to $H^2(G, \mathcal{O}_L^\times)$.

$\Rightarrow \mathrm{ram}(D_0) = 0$.

$$[(K/F, \sigma, t)] \in H^2(K/F, K^*)$$

$$K \oplus Ku \oplus Ku^2 \oplus \cdots \oplus Ku^{e-1} \qquad u^e = t$$

$$ux = \sigma(x)u$$

$$\tau \in G \longmapsto u_\tau$$

$$u_{\sigma^i} = u^i \qquad u_\tau u_\gamma = c(\tau,\gamma)u_{\tau\gamma}$$

$$u_{\sigma^i}u_{\sigma^j} = u^i u^j = \begin{cases} i+j<e & u^{i+j} \\ i+j \geq e & t\,u^{i+j-e} \end{cases}$$

$$c(\sigma^i,\sigma^j) = \begin{cases} 1 & \text{if } i+j<e \\ t & \text{if } i+j\geq e. \end{cases}$$

$$v(c(\sigma^i,\sigma^j)) = \begin{cases} 0 & \text{if } i+j<e \\ 1 & \text{if } i+j\geq e \end{cases}$$

$$\chi \in \mathrm{Hom}(\mathrm{Gal}(K/F), \mathbb{Q}/\mathbb{Z}) = H^1(\mathrm{Gal}(K/F), \mathbb{Q}/\mathbb{Z})$$

$$\sigma \longmapsto [1/e]$$

$$\sigma^i \longmapsto [i/e]$$

$$\tilde{\chi}(\sigma^i) = i/e \qquad \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$$

$$C^1 \qquad Z^1$$

$$\partial\tilde{\chi}(\sigma^i,\sigma^j) = \tilde{\chi}(\sigma^j) - \tilde{\chi}(\sigma^{i+j}) + \tilde{\chi}(\sigma^i) \qquad \tilde{\chi} \qquad \chi$$

$$Z^2 \longrightarrow B^2$$

$$= \begin{cases} 0 & \text{if } i+j<e \\ \end{cases}$$

$$= \begin{cases} 0 & \text{if } i+j < e \\ 1 & \text{if } i+j \geq e \end{cases}$$

Thms Albert-Brauer-Hasse-Noether theorem (1/2 of it)

$$Br(\mathbb{Q}_p) \xrightarrow{\text{ram}} \text{Hom}(Gal(\mathbb{F}_p), \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

is an isom. and

$$Br(\mathbb{Q}) \longrightarrow \bigoplus_p \mathbb{Q}/\mathbb{Z} \quad \text{is injective.}$$

$$0 \to Br(\mathbb{Q}) \longrightarrow \Big(\bigoplus_p \mathbb{Q}/\mathbb{Z}\Big) \oplus \underset{\substack{\uparrow \\ Br(\mathbb{R})}}{\overset{\substack{\mathbb{Q}/\mathbb{Z} \\ \cup}}{\mathbb{Z}/2}} \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \to 0$$

exact.

Next time: Survey of:

6

- Hom Vars / Severi-Br varieties 2 1/2 2
- ~~Br-Manin Obstructions & rat'l pts 2~~
- ~~Luroth Problem~~
- ~~Inverse~~ Gal thy "Generic gal exts" 3 2
- $\boxed{\text{Gerbes \& moduli problems. 2 1/2}}$ 4