Tensor products:

R, S, T rings $\quad {}_R M_S \quad {}_S N_T \quad {}_R P_T$

$$\varphi : M \times N \longrightarrow P$$

we say that φ is R-S-T linear if

1. $\forall\, n \in N \quad m \longmapsto \varphi(m,n)$ is left R-mod hom.
2. $\forall\, m \in N \quad n \longmapsto \varphi(m,n)$ is r. T-mod hom
3. $\varphi(ns, m) = \varphi(n, sm)$

<u>Def</u> Given ${}_R M_S$, ${}_S N_T$, we say that a bimod ${}_R P_T$ together w/ an R-S-T linear map $M \times N \to P$ is a tensor product of M & N over S if

$\forall\, M \times N \to Q$ R-S-T linear, $\exists!\ P \to Q$

s.t.
$$\begin{tikzcd} M \times N \arrow[rr] \arrow[dr] & & Q \\ & P \arrow[ur] & \end{tikzcd}$$
commutes.

<u>Def</u> $M \otimes_S N$ = the quotient of the free ab gp gen by $M \times N$ by the subgp $(ms, n) - (m, sn)$

$(m_1+m_2, n) - (m_1, n) - (m_2, n)$

$(m, n_1+n_2) - (m, n_1) - (m, n_2)$

<u>Notation:</u> $(m,n) \leftrightarrow m \otimes n$

Given M_R, ${}_RN \rightsquigarrow {}_{\mathbb{Z}}M_R$, ${}_RN_{\mathbb{Z}}$ $\quad M \otimes_R N$ $\mathbb{Z}$-mod

In the case where R comm.

leftmodules have r. module structure (in the stupid)

In this way $M_R \otimes_R {}_RN$ c — R-mod structure

R comm. R-R-R linear $\leftrightarrow$ R bilinear.

<u>Case of $\otimes$ over fields</u>

<u>Prop</u>: If V, W v.spaces over a field F, w/ bases. $\{v_i\}, \{w_j\}$, then $V \otimes W$ is a vector space w/ basis $\{v_i \otimes w_j\}$

<u>Pf</u>: clearly these span

$$\sum_k a_k \otimes b_k = \sum_k \left(\sum_i a_{i,k} v_i\right) \otimes \left(\sum_j b_{j,k} w_j\right)$$

$a_k \in V$ $b_k \in W$

$$= \sum_{i,j,k} a_{i,k} b_{j,k} v_i \otimes w_j$$

$$\varphi_{k,\ell} : V \times W \longrightarrow F$$
$$\left(\left(\sum \alpha_i v_i\right), \left(\sum \beta_j w_j\right)\right) \longmapsto \alpha_k \beta_\ell$$

bilinear.

$$\sum \gamma_{i,j} v_i \otimes w_j \longrightarrow \gamma_{k,\ell} \qquad \square.$$

If V/F vector space L/F field ext $L \otimes_F V$ is an L-space, with basis $\{1 \otimes v_i\}$ where $\{v_i\}$ basis for V.

Given a linear trans $T : V \longrightarrow W$

$$L \otimes T : L \otimes V \longrightarrow L \otimes W$$

$$L \otimes T\,(x \otimes v) = x \otimes Tv$$

identify the bases of V & $L \otimes V$, we see that T & $L \otimes T$ have the "same" matrix.

$$L \otimes (\ker T) = \ker (L \otimes T) \qquad \text{coker, im, etc.}$$

via Gaussian elim, etc.

Tensor products of algebras

$\Rightarrow A \otimes B$ is naturally

A, B F-algebras $\Rightarrow$ $A \otimes B$ is naturally an F-algebra via

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

Inside $A \otimes B$, $A \otimes 1$, $1 \otimes B$ are subalgebras $\cong$ to A & B & $A \otimes 1$ commutes w/ $1 \otimes B$

<u>Prop</u> Suppose A, B F-algebras, then for any F-algebra C, there is a bijection between

$$\{\text{hom. } A \otimes B \to C\} \longleftrightarrow \left\{ \begin{array}{l} \text{homs } A \to C \\ \phantom{\text{homs }} B \to C \end{array} \text{ s.t. images of } A \text{ \& } B \text{ commute in } C \right\}$$

<u>Pf:</u> $A \otimes B \to C$ get $A \otimes 1 \to C$, $1 \otimes B \to C$, with $A \otimes 1 \cong A$, $1 \otimes B \cong B$

which commute in C since $A \otimes B$ generated as an alg. by $A \otimes 1$, $1 \otimes B$

given $A \xrightarrow{\varphi_1} C$ $B \xrightarrow{\varphi_2} C$ define

$$A \otimes B \to C$$
$$a \otimes b \mapsto \varphi_1(a) \cdot \varphi_2(b)$$

$\triangleright$

A, B F-algebras given ${}_A M_B$ get homs

$$A \longrightarrow \mathrm{End}_F M$$
$$B^{op} \longrightarrow \mathrm{End}_F M$$

s.t. images of A & B^{op} commute

$(am)b = a(mb) \Rightarrow$ get a map
$$A \otimes B^{op} \longrightarrow \mathrm{End}_F M$$

$\Rightarrow$ get left $A \otimes B^{op}$-module structure on M.

Natural equiv. of cats between A-B bimods & left $A \otimes B^{op}$-modules.

Commutators

Notation given A/F algebra, $\Lambda \subset A$ subset
$$C_A(\Lambda) = \{a \in A \mid a\lambda = \lambda a \text{ all } \lambda \in \Lambda\}$$

$C_A(A) = Z(A)$

Suppose M a right A module. get a hom
$$A^{op} \longrightarrow \mathrm{End}_F(M).$$

If we let $C = C_{\mathrm{End}_F(M)}(A^{op}) = \mathrm{End}_A(M)$
to preserve sanity, we will regard M as a left C-module.
this gives M the structure of a $C-A$ bimodule.

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this gives M the structure ...

Theorem (DCT - warmup)
Let B be an F-algebra, M a faithful semisimple right B module, f.dim'l $/F$. let $E = \mathrm{End}_F(M)$ $C = C_E(B^{op})$.
Then $B^{op} = C_E(C) = C_E(C_E(B^{op}))$

Pf: let $\varphi \in C_E(C)$. Choose $m_1, \ldots, m_n$ a basis for M/F.
write $N = \bigoplus^n M \ni w = (m_1, \ldots, m_n)$. M semisimple $\Rightarrow N$ ssimple.
$\Rightarrow N = wB \oplus N'$ some N', set $\pi : N \to N$ projection (r. B-mod map)

$$\pi: N \to wB \to N$$

$$\pi \in \mathrm{End}_B(N) = M_n(\mathrm{End}_B M) = M_n(C_{\underbrace{\mathrm{End}_F M}_{E}}(B^{op})) = M_n(C)$$

Set $\varphi^{\oplus n} : N \to N$ doing φ on each entry

$$w\varphi^{\oplus n} = (\pi w)\varphi^{\oplus n} = \pi(w\varphi^{\oplus n}) \in wB$$

$\pi w = wb$ some b.

entries of π (in C) commute w/ $\varphi \in$ in $C_E(C)$

general principle:
$M_n(\text{Stuff})$ commutes w/ "scalar matrices" whose entries commute w/ stuff

$$\begin{bmatrix} a_{ij} \end{bmatrix}, \begin{bmatrix} b & & 0 \\ & \ddots & \\ 0 & & b \end{bmatrix}$$ commute if b's comm. w/ a_{ij}'s.

$\begin{bmatrix} a_{ij} \end{bmatrix}, \begin{bmatrix} 0 & \\ & b \end{bmatrix}$ commute iff ... a_{ij}'s.

<u>Goal</u>: Prove: if A is a CSA/F then $A \otimes_F A^{op} \simeq \mathrm{End}_F(A)$

notice that A is an A-A bimod

$$\Rightarrow A \otimes_F A^{op} \to \mathrm{End}_F(A)$$

Suppose $\{a_i\}$ a basis for A ($\&$ A^{op})

$$\sum c_{ij} a_i \otimes a_j \xrightarrow{?} 0 \text{ in } \mathrm{End}(A).$$

$A, B \subset E$

commuting subalgebras

<u>Lem</u>✩

A CSA

$a_i \in A$ indep $b_j \in B$ indep

$\Rightarrow a_i b_j$ indep in E

E is an A-A bimod $\Rightarrow$ $A \otimes A^{op}$ left mod

E is a right B-module $\Rightarrow$ $A \otimes A^{op}$-B bimodule.

A is a CSA $\Rightarrow$ it is a simple $A \otimes A^{op}$-mod

$$\mathrm{End}_{A\otimes A^{op}}(A) = F = Z(A)$$

$$C_{\mathrm{End}_F(A)}(C_{\mathrm{End}_F(A)}(A\otimes A^{op})) = C_{\mathrm{End}_F(A)}(F)$$

$\simeq \mathrm{End}_F(A)$

"lie"

image of this.

Really: $$C_{\mathrm{End}_F(A)}\left(C_{\mathrm{End}_F(A)}(\mathrm{im}(A\otimes A^{op}))\right) = C_{\mathrm{End}_F(A)}(F)$$

$\| $

$\mathrm{End}_F(A)$

$C(C(\mathrm{im}))$ = everything

DCT $\Rightarrow$ im = everything. $A\otimes A^{op} \to \mathrm{End}_F(A)$ surjective.

by dim count, its an $\simeq$.

So A CSA $\Rightarrow A\otimes A^{op} \simeq \mathrm{End}_F(A) = M_n(F)$

$n = \dim_F A$

Prop A CSA $\Leftrightarrow \exists B$ s.t. $A\otimes B \simeq M_n(F)$

Pf $\Rightarrow$ ✓

$\Leftarrow$ if $A\otimes B \simeq M_n(F)$, note $M_n(F)$ are central simple.

if $I \triangleleft A \Rightarrow I\otimes B \triangleleft M_n(F)$ by dim count, if I nontrivial, so is $I \otimes B$.

$\Rightarrow A$ simple.

if $1 \otimes b = a \otimes 1$

$\Rightarrow a, b \in F$

(by $\otimes$ of vector space basis stuff)

$$Z(A) = C_{M_n(F)}(A) \cap A = B \cap A = F$$

Know $B \subset C_{M_n(F)}(A)$

$$\text{lem}\,\ast \Rightarrow A \otimes C_{M_n(F)}(A) \hookrightarrow M_n(F)$$

$$A \otimes B \simeq M_n(F)$$

$$\Rightarrow B = C_{M_n(F)}(A)$$

<u>Prop</u> A CSA$/F \Longleftrightarrow \forall L/F$ field ext s.t. $L \otimes_F A$ CSA$/L$

$$\Updownarrow$$

$$\bar{F} \otimes_F A \simeq M_n(\bar{F})$$

<u>Pf:</u>

$\forall L/F$

$$A \text{ CSA} \Rightarrow A \otimes A^{op} \simeq M_n(F) \Rightarrow (A \otimes_F A^{op}) \otimes_F L \simeq M_n(L)$$

$$\|$$

$$(A \otimes L) \otimes_L (A^{op} \otimes L)$$

$$\Downarrow$$

$A \otimes L$ CSA all L

$$\Downarrow$$

$A \otimes_F \bar{F}$ CSA

$$\Downarrow$$

W-A : $A \otimes_F \bar{F} \simeq M_n(D)$

$Z(A \otimes \overline{F}) = Z(A) \otimes \overline{F}$.
(kernel of linear map)

$D/\overline{F}$ f.dim'l
div alg

all $d \in D^*$, $\overline{F}[d]$ finite $/\overline{F}$

A simple since otherwise
$I \otimes \overline{F} \triangleleft A \otimes \overline{F} = M_n(\overline{F})$

finite field ext $\Rightarrow d \in \overline{F} \Rightarrow D = \overline{F}$

$A \otimes_F \overline{F} \simeq M_n(\overline{F})$

<u>Def</u> If A is a CSA,

$\overline{F} \otimes A \simeq M_n(\overline{F})$ n^2

$\deg(A) = \sqrt{\dim_F A}$

$A \simeq M_m(D)$

$\mathrm{ind}(A) = \deg(D)$

$Z(D) = F$ $\Rightarrow D$ CSA
(central division algebra)

"Schur index"

D = "underlying division algebra"

$D = \mathrm{End}_{\text{right } A}(P)$ P simple r. A-mod.

$\dim_F A = m^2 \dim_F D$ $\deg A = m \cdot \deg D = m \,\mathrm{ind}\, A$

$(\mathrm{ind} A \mid \deg A)$

<u>Brauer equivalence</u>

CSA's A, B we say $A \sim B$ iff $\exists\, r, s$ s.t.

$M_r(A) \simeq M_s(B)$

$$M_r(M_n(D_A)) \simeq M_s(M_m(D_B))$$
$$\Rightarrow D_A \cong D_B$$

$A \sim B$ iff underlying div. algebras are $\simeq$.

<u>Observation</u>:

if $A, B/_F$ CSA's $\Rightarrow A \otimes_F B$ also CSA

$$(A \otimes B) \otimes \bar{F} = (A \otimes \bar{F}) \otimes_{\bar{F}} (B \otimes \bar{F}) = M_n(\bar{F}) \otimes_{\bar{F}} M_m(\bar{F})$$
$$= M_n(M_m(\bar{F}))$$
$$= M_{nm}(\bar{F}) \quad \checkmark$$

<u>Def</u>: $Br(F)$ is the group of Br. eq. classes of CSA's/F

w/ operation $\quad [A] + [B] = [A \otimes_F B]$

id. elmt $[F]$

$$[A] + [A^{op}] = [A \otimes A^{op}] = [M_{\dim_F A}(F)] = [F]$$

<u>Def</u> The exponent of A (or period of A) is order $[A]$ in $Br(F)$.

Fact: per $A \mid$ ind A