Lecture 4: Crossed Products

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Last time(s)

DCT warmup 1: If $B \subset \mathrm{End}_F(V)$ B simple

Then $C_{\mathrm{End}_F(V)}(C_{\mathrm{End}_F(V)} B) = B$

DCT warmup 2: If $A \simeq B \otimes C$ all CSA/F $\Rightarrow$

$C = C_A(B)$

Noether-Skolem: If $B, B' \subset A$ CSA/F, B simple, $\psi : B \xrightarrow{\sim} B'$

$\Rightarrow \exists a \in A^\times$ s.t.

$\psi(b) = aba^{-1}$

DCT warmup 3: If $B \subset A$ CSA/F then

1. $C_A(B)$ is a CSA/F
2. $A = B\, C_A(B) \simeq B \otimes C_A(B)$

Pf: $2 \Rightarrow 1$ A simple $\Rightarrow C_A(B)$ simple

$1 \otimes Z(C_A(B)) \hookrightarrow Z(A) = F \Rightarrow C_A(B)$ central.

to prove 2, always have the map

$B \otimes C_A(B) \longrightarrow A$

is it an iso? WLOG, $F = \bar{F}$ in particular,

$B = M_n(F)$, $A = End(V)$ B simple $\Rightarrow \exists!$ simple module

F^n is one

$B \subset A \Rightarrow V$ a B mod $\Rightarrow V = (F^n)^m$

$$A = M_{nm}(F) = M_m(M_n(F))$$

$$C_A(B) = C_{M_m(M_n(F))}(M_n(F)) = M_m(Z(M_n(F)))$$

block sclars $= M_m(F)$

$$M_n(F) \otimes M_m(F) \simeq M_{mn}(F) \quad \square.$$

<u>Theorem</u> Full-On DCT. $B \subset A$ A a CSA$/F$ B simple

then
1. $C_A(B)$ simple
2. $(\dim_F B)(\dim_F C_A(B)) = \dim_F A$
3. $C_A(C_A(B)) = B$
4. if B is a CSA$/F \Rightarrow A \simeq B \otimes C_A(B)$

<u>Pf.</u> 4✓

3: $B \hookrightarrow A \hookrightarrow A \otimes A^{op} = End_F(A)$

know by warmup 1 $B = C_{End_F(A)}(C_{End_F(A)} B)$

$$C_{End_F(A)}(B) = C_{A \otimes A^{op}}(B) = C_A(B) \otimes A^{op}$$

$(\sum a_i \otimes a_i') b = b \sum a_i \otimes a_i'$ a_i' indep

$$\left(\sum a_i \otimes a_i'\right) b = b \sum a_i \otimes a_i'$$

a_i' indep

$$= \sum (a_i b - b a_i) \otimes a_i'$$

$$C_{A\otimes A^{op}}\left(C_A(B) \otimes A^{op}\right) = C_A(C_A(B)) \otimes 1$$

✓

$$\sum a_i \otimes a_i'$$

2. $C_A(B)$ $\quad$ B simple $\Rightarrow$ CSA$/L$ $\quad$ $L = Z(B)$

$$L \subset B \subset A \subset A \otimes A^{op} = \mathrm{End}_F A$$

Note that A is a (left) L-v.space,

B acts on A as L-linear maps

$$B \subset \mathrm{End}_L(A) \subset \mathrm{End}_F(A)$$

$$C_{A\otimes A^{op}}(B) = C_A(B) \otimes A^{op}$$

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$L \subset B \Rightarrow C_{A\otimes A^{op}}(B)$ acts on A via L-linear maps

$$C_{A\otimes A^{op}}(B) = C_{\mathrm{End}_F(A)}(B) = C_{\mathrm{End}_L(A)}(B)$$

is centralizer of a CSA/L in another ⇒ simple.

$$B \subset \mathrm{End}_L(A)$$

$$\mathrm{End}_L(A) = B \otimes_L C_{\mathrm{End}_L(A)}(B)$$

$$= B \otimes_L C_{\mathrm{End}_F(A)}(B)$$

$$\dim_L(\mathrm{End}_L(A)) = \dim_L(A)^2 = \left(\frac{\dim_F A}{[L:F]}\right)^2$$

$$\dim_L B = \frac{\dim_F B}{[L:F]} \qquad \dim_L C_{\mathrm{End}_L(A)}(B) = \frac{\dim_F C_{\mathrm{End}_L(A)}(B)}{[L:F]}$$

$$\parallel$$

$$\frac{\dim_F C_{\mathrm{End}_F(A)} B}{[L:F]} = \frac{\dim_F C_{A \otimes A^{op}}(B)}{[\ ]}$$

$$\parallel$$

$$\frac{(\dim_F C_A(B)) \dim_F A}{[L:F]} = \frac{\dim_F C_A(B) \otimes A^{op}}{[L:F]}$$

$$\frac{(\dim_F A)^2}{[L:F]^2} = \frac{\dim_F B}{[L:F]} \left(\frac{\dim_F C_A(B)\ \dim_F A}{[L:F]}\right) \checkmark$$

$\square$

Suppose A csa$/F$ & $E \subset A$ max'l subfield.

i.e. $[E:F] = \deg A$ & E/F is G-Galois.

In this case, if $\sigma \in G$, $\exists\, u_\sigma \in A^*$ s.t. $u_\sigma x u_\sigma^{-1} = \sigma(x)$ for $x \in E$.
(Noether-Skolem)

We'll show: $A = \bigoplus_{\sigma \in G} E u_\sigma$

u_σ "N-S elmts"

Claim: u_σ indep over E (on left)

if not, choose min'l dependence relation:

$$\sum x_\sigma u_\sigma = 0$$

$$0 = \sum x_\sigma u_\sigma y = \sum x_\sigma \sigma(y) u_\sigma$$ all $y \in E \setminus F$

$\Rightarrow \lambda x_\sigma = x_\sigma \sigma(y)$ all σ some fixed λ

i.e. $\sigma(y) = \lambda$ all σ $\Rightarrow y \in F$

thanks, Artin

$\Rightarrow$ by dim count $A = \bigoplus E u_\sigma$.

if u_σ & v_σ are both N-Skolem for $\sigma \in G$

$\Rightarrow u_\sigma v_\sigma^{-1} x = x u_\sigma v_\sigma^{-1}$ $\Rightarrow$ $u_\sigma v_\sigma^{-1} \in C_A(E) = E$
$x \in E$ (DCT)

$\Rightarrow v_\sigma = \lambda_\sigma u_\sigma$ some $\lambda_\sigma \in E^*$

conversely, such a v_σ is a NS elmt for σ

Notice: $u_\sigma u_\tau$ $\quad u_{\sigma\tau}$ both are NS for $\sigma\tau$

$$\Rightarrow u_\sigma u_\tau = c(\sigma,\tau) u_{\sigma\tau} \qquad c(\sigma,\tau) \in E^\times$$

Associativity: $u_\sigma(u_\tau u_\gamma) = (u_\sigma u_\tau) u_\gamma$

$$u_\sigma(c(\tau,\gamma)u_{\tau\gamma}) \qquad (c(\sigma,\tau)u_{\sigma\tau})u_\gamma$$

$$\sigma(c(\tau,\gamma))\, u_\sigma u_{\tau\gamma} \qquad c(\sigma,\tau)\, u_{\sigma\tau} u_\gamma$$

$$\sigma(c(\tau,\gamma))\, c(\sigma,\tau\gamma)\, u_{\sigma\tau\gamma} \qquad c(\sigma,\tau)\, c(\sigma\tau,\gamma)\, u_{\sigma\tau\gamma}$$

$$\Rightarrow \quad c(\sigma,\tau)\, c(\sigma\tau,\gamma) = c(\sigma,\tau\gamma)\, \sigma(c(\tau,\gamma))$$

2-cocycle condition! fcn $c: G\times G \to E^\times$ is a 2 cocycle if

$$\begin{pmatrix} \sigma\ \tau\ \gamma \\ (\sigma,\tau)\gamma - (\sigma,\tau\gamma) + (\sigma\tau,\gamma) - \sigma(\tau,\gamma) \end{pmatrix}$$

Def: If E/F is G Galois, $c: G\times G \to E^\times$ a 2-cocycle, then define (E,G,c) to be

$\bigoplus E u_\sigma$ w/ mult. defined by

$x, y \in E \qquad (x u_\sigma)(y u_\tau) = x\,\sigma(y)\,c(\sigma,\tau)\,u_{\sigma\tau}$

<u>Prop</u> $A := (E, G, c)$ as above is a CSA/F.

<u>Pf:</u> simple?

If $A \twoheadrightarrow B$ then $E \hookrightarrow B$ (simplicity of E)

and $u_\sigma \longrightarrow v_\sigma \in B$ are N-S in B for E

$\Rightarrow$ indep in B $\Rightarrow$ injective $\Rightarrow$ $\ker = 0$ ✓

central? $Z(A) \subset C_A(E) = E$

$(\sum x_\sigma u_\sigma) y = y (\sum x_\sigma u_\sigma)$ all $y \in E$

$\Rightarrow \sum (x_\sigma \sigma(y) - y x_\sigma) u_\sigma = 0$

$\Rightarrow$ either $x_\sigma = 0$

or $\sigma y = y$ all y

$\Rightarrow \sigma = 1$

$Z(A) \subset C_A\{\{u_\sigma\}_{\sigma \in G}\} \cap E = F$

$\square$.

<u>Q:</u> When is $(E, G, c) \overset{\varphi}{\simeq} (E, G, c')$?

NS $\Rightarrow$ isom. preserves E. $\varphi(E) = E$

$\varphi(u_\sigma)$ is a NS in (E, G, c')

$(E, G, c) = \bigoplus E u_\sigma \qquad (E, G, c') = \bigoplus E u'_\sigma$

$\varphi(u_\sigma) = x_\sigma u'_\sigma \qquad x_\sigma \in E^*$

Hom $\Rightarrow$ $\varphi(u_\sigma u_\tau) = \varphi(u_\sigma)\varphi(u_\tau)$

$$\begin{aligned}\varphi(u_\sigma u_\tau) &= \varphi(c(\sigma,\tau)u_{\sigma\tau}) = c(\sigma,\tau)\, x_{\sigma\tau} u'_{\sigma\tau}\\ \varphi(u_\sigma)\varphi(u_\tau) &= (x_\sigma u'_\sigma)(x_\tau u'_\tau) = x_\sigma \sigma(x_\tau)\, c'(\sigma,\tau)\, u'_{\sigma\tau}\end{aligned}$$

$$\Rightarrow \quad c(\sigma,\tau)\, x_{\sigma\tau} = x_\sigma \sigma(x_\tau)\, c'(\sigma,\tau)$$

i.e. $\quad c(\sigma,\tau) = x_\sigma \sigma(x_\tau) x_{\sigma\tau}^{-1} c'(\sigma,\tau)$

some elmts $x_\sigma \in E^\times$ ~~each~~ $\sigma \in G$.

We say c, c' are cohomologous if $\exists\, b : G \to E^\times$

s.t. $c(\sigma,\tau) = b(\sigma)\sigma(b(\tau))b(\sigma\tau)^{-1} c'(\sigma,\tau)$

Set $B^2(G,E^\times) = \{ f : G\times G \to E^\times$ s.t. $f = b(\sigma)\sigma(b(\tau)) b(\sigma\tau)^{-1}$ some $b : G \to E^\times \}$

$Z^2(G,E^\times) = \{ f : G\times G \to E^\times \mid 2 \text{ cocycles} \}$

these are groups via pt.-wise mult.

Def $\quad H^2(G,E^\times) \equiv \dfrac{Z^2(G,E^\times)}{B^2(G,E^\times)}$

Prop $\quad H^2(G,E^\times)$ is in bijection w/ Isom. classes of CSA/F s.t. $E \subset A$ max'l.

Idempotents

$e \in A \quad e^2 = e.$ $\longrightarrow$ central

$e(1-e) = e - e^2 = 0$

$e + (1-e) = 1$

$(1-e)^2 = 1 - 2e + e^2 = 1 - e.$

$A = A \cdot 1 = A(e + (1-e))$

$= \underset{\substack{\Psi \\ e}}{Ae} \times \underset{\substack{\Psi \\ (1-e)}}{A(1-e)}$

$(ae)(be) = (aeb)e$

$\downarrow$ nat