Lecture 5: second and first Galois cohomology groups

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Last time:

$Br(E/F) \longleftrightarrow$ eq classes. of CSA w/ E max'l subfield

(E/F G-Galois)

$$Br(E/F) \simeq H^2(G, E^\times) = \frac{Z^2(G,E^\times)}{B^2(G,E^\times)}$$

$(E, G, c) \longleftarrow c$

$G \times G \xrightarrow{c} E^\times$

$c(\sigma,\tau)c(\sigma\tau,\gamma) = c(\sigma,\tau\gamma)\sigma(c(\tau,\gamma))$

$b(\sigma\tau)b(\sigma)^{-1}\sigma(b(\tau)^{-1})$

Goal: relate splitting fields to max'l subfields.

Def: We say E/F splits A if $A \otimes_F E \simeq M_n(E)$

Know: always have splitting fields – $\bar{F}$

in fact, there splitting fields which are finite extensions.

Lemma If A CSA/F, $E \subset A$ subfield, then

$C_A(E) \sim A \otimes_F E$.

Pf: $A \otimes E \hookrightarrow A \otimes A^{op} = End_F A$

$$C_{End_F(A)}(E) = A \otimes_F C_{A^{op}}(E) = A \otimes_F E \otimes_E C_{A^{op}}(E)$$

E-algebra $= (A \otimes E) \otimes_E (C_{A^{op}}(E))$

$\cong C_A(E)^{op}$

$\mathrm{End}_{r\text{-}E}(A)$

$\shortparallel$ split E-alg

$[A \otimes E] - [C_A(E)] = 0$

$\square$.

<u>Cor</u> If $E \subset D$, D CDA$/F$ then $(\mathrm{ind}\, D \otimes E)\,[E:F] = \mathrm{ind}\, D$

$\quad\quad C_D(E)$

<u>Pf</u>: $\underbrace{\dim_F C_D(E)}\,[E:F] = \dim_F D$

$(\dim_E C_D(E))\,[E:F]$

$\Downarrow$

$\deg C_D(E)^2 \cdot [E:F]^2 = (\deg D)^2$

$(\deg C_D E)\,[E:F] = \deg D = \mathrm{ind}\, D$

$\mathrm{ind}\, C_D E$

$\shortparallel$

$\mathrm{ind}\, D \otimes E$

$\square$

<u>Remark</u>: if $E \subset A$ is a max'l subfield, then $A \otimes E$ is split

why? because $A \otimes E \sim C_A(E) = E$ ✓

<u>Prop</u>: If A CSA$/F$, $E \otimes A \simeq M_n(E)$, and $[E:F] = \deg A = n$, then E is isom. to a (max'l) subfield of A.

<u>Pf</u>: $E \hookrightarrow \mathrm{End}_F E = M_n(F) \hookrightarrow A \otimes M_n(F)$

$C_{A \otimes M_n(F)}(E) \simeq M_n(E)$

$$C_{A\otimes M_n(F)}(E) \simeq M_n(E)$$

$$\wr \qquad\qquad \parallel$$

$$(A\otimes M_n(F))\otimes_F E \qquad \underbrace{E\otimes M_n(F)}_{B}$$

$$A\otimes M_n(F) \xleftarrow{\varphi} E\otimes M_n(F)$$

$$\text{"}B\text{"} \longleftarrow M_n(F) \subset E\otimes M_n(F)$$

replace φ by φ comp w/ inner aut so that B is $1\otimes M_n(F)$.

so now note $E \subset E\otimes M_n(F)$

$\subset C_{E\otimes M_n(F)}(M_n(F))$

$\Rightarrow \varphi(E) \subset C_{A\otimes M_n(F)}(M_n(F)) = A\otimes 1$ $\square$.

Cor Let A/F be a csA/F then $[A] \in Br(E/F)$ some E/F Galois.

Pf: write $A = M_m(D)$ $[A]=[D]$ wLOG A division.

Know D has a max'l sep. subfield $L\subset D$

Let E/F = Gal. closure of L/F.

$E \hookrightarrow End_L(E) = M_{[E:L]}(L)$

$D\otimes_F M_{[E:L]}(F) \supset L\otimes_F M_{[E:F]}(F) = M_{[E:L]}(L) \supset E$

$D \otimes_F M_{[E:L]}$

$$\deg = \deg D \cdot [E:L] = [E:F]$$
$$[L:F]$$

$\Rightarrow E \cong$ max'l subfield of $D \otimes M_{[E:L]}(F)$

$\Rightarrow [A] \cong [D] \in Br(E/F)$ $\square$

Galois Descent.

A is a CSA $\Leftrightarrow A \otimes E \cong M_n(E)$ some E/F Galois.

A is a "twisted form" of a matrix algebra.

Def Given an alg A/F, we say that B/F is a twisted form of A if $A \otimes_F E \cong B \otimes_F E$ some E/F separable (Galois)

"**Def**" Given any "algebraic structure" A/F we say B/F is a tw. form if $A \otimes_F E \cong B \otimes_F E$ some E as above.

descent = going from $E \rightsquigarrow F$.

$E^G = F$. $\quad G = Gal(E/F)$

Idea: given $A \otimes E$, G acts on the E-part
$$\sigma(a \otimes x) \equiv a \otimes \sigma x$$

and invariants give A.

Fix E/F a G-Galois extension.

<u>Def</u> A semilinear action of G on an E-vector space V is an action of G on V (as F-linear transformations) s.t. $\sigma(xv) = \sigma(x)\sigma(v) \quad \forall x \in E, v \in V$.

<u>Thm</u> there is an equivalence (of cats)

$$\underline{F\text{-vector spaces}} \underset{-\otimes_F E}{\overset{(-)^G}{\rightleftarrows}} \underline{E\text{-v.spaces w/ semilinear actions}}$$

If V is an E-space w/ semilinear action, get an action of (left-module) $(E,G,1)$ on $V \longleftarrow \bigoplus E u_\sigma \quad u_\sigma u_\tau = u_{\sigma\tau}$, $u_\sigma x = \sigma(x) u_\sigma$

via $(xu_\sigma)(v) = x\sigma(v)$

$(xu_\sigma)(yu_\tau)(v) = xu_\sigma(y\tau(v)) = x\sigma(y)\sigma\tau(v)$

$= (x\sigma(y)u_\sigma u_\tau)(v)$

$= x\sigma(y)u_{\sigma\tau}(v) = x\sigma(y)\sigma\tau(v)$

E-space $\qquad E \cap (E,G,1)$

Actually, a semilinear action on V is a $(E,G,1)$ mod structure.

E-space; $\sigma(v)$; $u_\sigma \cdot v$

$(E,G,1)$ has a unique simple module : E

$\left(V \text{ semilinear} \longleftrightarrow V \simeq E^n \right)$

If V is semilinear space, so a $(E,G,1)$-module,

then $V^G \simeq E' \otimes_{(E,G,1)} V$ where E' is the unique simple right $(E,G,1)$-module.

? hmm...

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if V, W are semilinear spaces, then a semilinear hom is

$\varphi : V \longrightarrow W$ F linear s.t.

$\varphi(\sigma v) = \sigma(\varphi(v))$

$V \longrightarrow V^G$

$W \otimes E \longleftarrow W$

$$\begin{array}{ccccc} W & \longrightarrow & W \otimes E & \longrightarrow & (W \otimes E)^G \\ \| & & \| & & \\ \oplus F e_i & & \oplus E e_i & & \oplus E^G e_i \\ & & & & \| \\ & & \xrightarrow{\sim} & & \oplus F e_i \end{array}$$

$$V = \oplus E e_i \longrightarrow \oplus E^G e_i = \oplus F e_i$$

$V = \bigoplus E e_i \rightarrow \bigoplus E^G e_i = \bigoplus F e_i \rightarrow \bigoplus (F \otimes_F E) e_i = \bigoplus E e_i$

$\rightarrow V^G \rightarrow V^G \otimes_F E$

if $\varphi : W \longrightarrow W'$ F-linear map

$\varphi \otimes E : W \otimes E \to W' \otimes E$

$$\begin{array}{ccc} a \otimes x & \longmapsto & \varphi(a) \otimes x \\ \downarrow \sigma & & \downarrow \sigma \\ a \otimes \sigma(x) & \longrightarrow & \varphi(a) \otimes \sigma(x) \end{array}$$

if $V \xrightarrow{\psi} V'$ is semilinear then ψ induces a map (via restriction)

to $V^G \xrightarrow{\psi^G} V'^G$

$$\begin{array}{ccc} v & \xrightarrow{\psi} & \psi(v) \\ \downarrow \sigma & & \downarrow \sigma \\ v & \xrightarrow{\psi} & \psi(v) \end{array}$$

if V, W are semilinear spaces, how should we define semilinear action on $V \otimes_E W$? via

$\sigma(v \otimes w) = \sigma(v) \otimes \sigma(w)$

$V = \bar{V} \otimes E \qquad W = \bar{W} \otimes E$

$V \otimes_E W = (\bar{V} \otimes E) \otimes_E (\bar{W} \otimes E) = (\bar{V} \otimes \bar{W}) \otimes E$

$(\bar{V} \otimes E) \otimes_E (\bar{W} \otimes E) \longleftrightarrow (\bar{V} \otimes \bar{W}) \otimes E$

E

$\sigma(b)$

$\}E$

$\bar{V} \otimes \bar{W}$

$\bar{v}\otimes\bar{w}\otimes ab \xrightarrow{\sigma} \bar{v}\otimes\bar{w}\otimes\sigma(a)\sigma(b)$

$(\bar{v}\otimes a)\otimes(\bar{w}\otimes b)$ $\qquad \sigma(\bar{v}\otimes\bar{w}\otimes x)$

$\downarrow\sigma$ $\qquad\qquad ''$

$(\bar{v}\otimes\sigma(a))\otimes(\bar{w}\otimes\sigma(b))$ $\qquad \bar{v}\otimes\bar{w}\otimes\sigma(x)$

(eq of cat w/ $\otimes$) ✓

<u>Defn</u> A semilinear action of G on an alg $A/_E$ is

$G \longrightarrow \mathrm{Aut}(A/F)$ s.t. $\sigma(xa) = \sigma(x)\sigma(a)$ all $x \in E$, $a \in A$

$\sigma(ab) = \sigma(a)\sigma(b)$

$\updownarrow$

$A\otimes A \longrightarrow A$ is semilinear.

Theorem of descent then says

<u>semilinear algs</u>$/E \longleftrightarrow$ <u>F-algebras</u>

A = interesting algebra. Find twisted forms of A.

if B is a tw. form

$$A\otimes E \xleftarrow[\sim]{\varphi} B\otimes E$$

can define new action

$$\sigma_B(\alpha) = \varphi((1\otimes\sigma)(\varphi^{-1}(\alpha)))$$

$\alpha \in A\otimes E$

compare: $\sigma^{-1}(\sigma_B(\_)) \in \mathrm{Aut}_{\underset{\uparrow}{E}}(A\otimes E)$

$$\sigma^{-1}(\sigma_B(x\alpha)) = \sigma^{-1}(\varphi\,\sigma(\varphi^{-1}(x\alpha)))$$
$$= \sigma^{-1}(\varphi\,\sigma(x\,\varphi^{-1}(\alpha)))$$
$$= \sigma^{-1}(\varphi\,\sigma(x)\,\sigma(\varphi^{-1}(\alpha)))$$
$$= x\,\sigma^{-1}\varphi\sigma\varphi^{-1}\alpha = x\,\sigma^{-1}(\sigma_B(\alpha))$$

also, $\overset{\varphi\sigma\varphi^{-1}}{\overset{\shortparallel}{\sigma_B}}\circ\sigma^{-1} \in \mathrm{Aut}_E(A\otimes E)$

$\sigma_B = a_\sigma\sigma$ $\qquad a_\sigma \in \mathrm{Aut}_E(A\otimes E)$

$$\sigma_B\tau_B = (\sigma\tau)_B$$
$$\Downarrow$$
$$a_\sigma\circ\sigma(a_\tau\tau(\alpha)) = a_{\sigma\tau}\,\sigma\tau(\alpha)$$
$$a_\sigma\,\sigma(a_\tau)\,\sigma\tau(\alpha)$$

$$a_{\sigma\tau} = a_\sigma\,\sigma(a_\tau)$$

1-cocycle

$a(\sigma)$ $\qquad a(\sigma\tau) = a(\sigma)\,\sigma(a(\tau))$

"crossed hom"

A then

Thm If B is a twisted form of A, then $\exists\, a : G \to \mathrm{Aut}_E(A\otimes E)$ a 1-cocycle & such that $B = (A\otimes E)_a^G$ where $(A\otimes E)_a = A\otimes E$ w/ new action via

$$\sigma_a(\alpha) = a_\sigma \sigma(\alpha)$$

& conversely, every such 1-cocycle gives a twisted form.

Pf: given a 1-cocycle $a : G \to \mathrm{Aut}(A\otimes E)$, let's check that action on $(A\otimes E)_a$ is semilinear.

$$\sigma_a \tau_a(\alpha) = (\sigma\tau)_a(\alpha) \quad \& \quad \sigma_a(x\alpha) = \sigma(x)\sigma_a(\alpha)$$

$$\begin{aligned}
\sigma_a\tau_a(\alpha) = a_\sigma \sigma(a_\tau \tau(\alpha)) &= a_\sigma \sigma(a_\tau)\,\sigma(\tau(\alpha)) \\
&= a_\sigma \sigma(a_\tau)\,\sigma\tau(\alpha) \\
&= a_{\sigma\tau}\,\sigma\tau(\alpha) = (\sigma\tau)_a(\alpha)
\end{aligned}$$

$$\begin{aligned}
\sigma_a(x\alpha) = a_\sigma \sigma(x\alpha) &= a_\sigma(\sigma(x)\sigma(\alpha)) \\
&= \sigma(x)\, a_\sigma(\sigma(\alpha)) = \sigma(x)\,\sigma_a(\alpha)
\end{aligned}$$

$\square$.

$$a_\sigma = \varphi\sigma\varphi^{-1}\sigma^{-1}$$

$$\begin{tikzcd} B\otimes E \arrow[r, bend left, "\varphi"] & A\otimes E \arrow[loop below, "b"] \end{tikzcd}$$

$$(b\varphi)\sigma((b\varphi)^{-1}(\sigma^{-1}(\alpha))))$$

$$(b\varphi)(\sigma((b\varphi)^{-1}(\sigma^{-1}(\alpha))))$$

$$= (b\varphi)(\sigma(\varphi^{-1}b^{-1})(\sigma^{-1}(\alpha))))$$

$$=$$

$$b\varphi\sigma\varphi^{-1}b^{-1}\sigma^{-1}\alpha$$

$$=$$

$$b\underbrace{\varphi\sigma\varphi^{-1}\sigma^{-1}}_{a_\sigma}\sigma b^{-1}\sigma^{-1}\alpha$$

$$b\, a_\sigma(\sigma b^{-1}\sigma^{-1})\alpha$$

we say that a_σ & a'_σ are cohomologous if

$a'_\sigma = b\, a_\sigma(\sigma b^{-1}\sigma^{-1})$ some $b \in \mathrm{Aut}(A\otimes E)$

$$\sum t_{ij}e_{ij} = T \circlearrowright E^n \circlearrowleft \sigma$$

$$\sigma T\sigma^{-1}\left(\sum v_i e_i\right)$$

$$= \sigma T \sum \sigma^{-1}(v_i)e_i$$

$$= \sigma\left(\sum \sigma^{-1}(v_i)\sum t_{ij}e_j\right)$$

$$= \sum v_i \sum \sigma(t_{ij})e_j$$

$\sum \sigma(t_{ij})e_{ij}$ acts on $\sum v_i e_i$!

eq. classes cohomologousness bijective w/ iso classes of similar actions & also therefore in bijection w/ twisted forms of A.

Def $H^1(G, \mathrm{Aut}(A \otimes E))$ is the set of cohomology classes.

$a_\sigma = 1$