# Étale Cohomology

1. What
2. Why
3. How

1. Étale cohomology

generalization of Galois cohomology

k field ⇝ rings ⇝ schemes ⇝ ?

$H^n(\mathrm{Gal}(k^{sep}/k), M) = H^n(k, M)$

$H^n_{\acute{e}t}(X, A)$ — $X$: scheme, $A$: "sheaf"

$X = \mathrm{Spec}\, k$   $H^n_{\acute{e}t}(\mathrm{Spec}\, k, A) = H^n(k, A(k^{sep}))$

Incorporates Betti cohomology w/ finite coefficients
(singular)

if $X/\mathbb{C}$ smooth   $H^n_{\acute{e}t}(X, \mathbb{Z}/n\mathbb{Z}) = H^n_{sing}(X_{\mathbb{C}}, \mathbb{Z}/n\mathbb{Z})$

2. Why?

- Conjectures of Weil in 1949 on $\# X(\mathbb{F}_{q^k})$

  $X/\mathbb{F}_q$

- Proposed these would follow from the existence of a "suitable cohomology"

$X \rightsquigarrow H^n(X, \text{--})$

Helps because of Lefschetz trace formula.

$f: X \longrightarrow X$ "count" # fixed pts

$$\#\text{fixed pts} = \sum_i (-1)^i \, tr(H^i(f, \mathbb{Q})) : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})$$

$\# X(\mathbb{F}_{q^k}) = \#$ fixed pts of power of Frobenius. (in $X(\overline{\mathbb{F}_q})$)

$(x_1, \ldots, x_N) \longmapsto (x_1^{q^k}, \ldots, x_N^{q^k})$

$\rightsquigarrow$ $\ell$-adic cohom. $\Leftarrow$ étale cohom.

3. How?

Two approaches to cohom. $\nearrow$ Derived functor of global sections

$\searrow$ Čech cohomology

in AG: $X$ scheme $\mathcal{F}$ q. coherent sheaf (Zariski top)

$\mathcal{F} \longmapsto \Gamma(\mathcal{F})$ left, not right exact.

$H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F})$

Čech philosophy: computations can be reduced to choosing an open cover,

computing cohom on opens & overlaps
+
combinatorics of open cov.

Good situation
only $H^0$'s are $\neq 0$.

- In alg top, generally can choose covs s.th. $H^i = 0$ on opens & overlaps

$H^0 = \mathbb{Z}, \mathbb{Q}$.

Soln to getting something close to Singular cohom:
generalizing notion of "open cov"
add "generalized open covs"

old $\quad \bigsqcup U_i \xrightarrow{f} X \qquad f^*(\text{interesting})$

new $\quad U \xrightarrow{f} X \qquad$ "trivial"

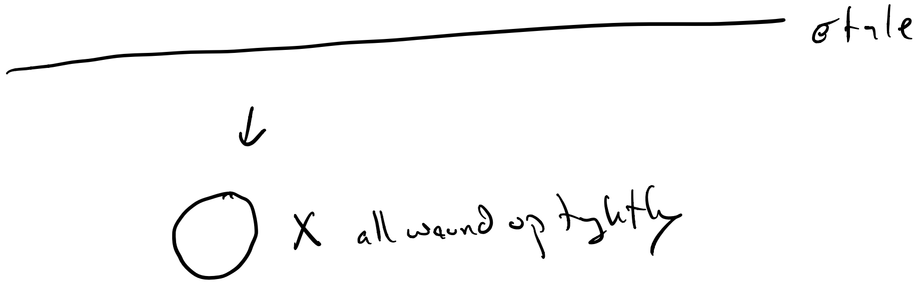
Grothendieck topology:
axiomatic formulation of which collections of morphisms can be thought of as "covs"

"interesting" — basically only worrying about "$H^1$ stuff"

enough to "trivializing loops"

answer: covering spaces.

étale = spread out flat, loose, slack, relaxed, flat

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<u>Step 1</u>: Grothendieck topologies, sheaves, cohomology (derived, Čech), computational tools (spectral seqs) ← Leray spectral seq.

<u>Step 2</u>: Étale cohomology, computations

<u>Step 3</u>: Key tools / theorems (Comparison, proper base chge)

$$H^n_{et}(X, \mathbb{Z}/\ell\mathbb{Z}) = H^n_{sg}(X_{\mathbb{C}}, \mathbb{Z}/\ell\mathbb{Z})$$