Grothendieck topologies

Abstraction of the category of open sets in a top space, together with the notion of covering.

Example: X top space ~~$\mathcal{C}$ = Cat w/ objects open sets $U \subset X$~~

$\mathcal{C}$ = objects are cont. maps $U \to X$ which are isoms w/ opens in X

$$U \xrightarrow{\ \sim\ } U' \subset X$$

instead of $\cap$ use fiber product.

$$\begin{array}{l} U \overset{\varphi}{\searrow} \\ \qquad\quad X \\ V \underset{\psi}{\nearrow} \end{array} \qquad U \times_X V = \{(u,v) \mid \varphi(u) = \psi(v)\} \cong \varphi(U) \cap \psi(V)$$

Def A Groth. top on a category $\mathcal{C}$ is a collection of "coverings", covering is a collection of morphisms $\{U_i \to U\}_{i \in I}$

such that

1.) if $\{U_i \to U\}$ is a covering and $V \to U$ any morphism then $\{U_i \times V \to V\}$ is a covering (and all these fiber products exist)

2.) if $\{U_i \to U\}_{i\in I}$, $\{V_j \to U_i\}_{j\in J_i}$ are covs

then $\{V_j \to U_i \to U\}$ is a cov.

3.) If $V \xrightarrow{\sim} U$ is an isom in $\mathcal{C}$, it is a cov.

<u>Language:</u> A category w/ a Grothendieck top is called a site.

if T a site. $cat(T)$ $\quad cov(T)$
$\mathcal{C}$ $\quad$ covs above.

<u>Def</u> A morphism of sites $f: T \to T'$ is a functor $cat(T) \xrightarrow{f} cat(T')$ s.t.

$\{U_i \to U\} \in cov(T) \Longrightarrow \{f(U_i) \to f(U)\} \in cov(T')$

and if $V \to U$ in $cat(T)$ then

$$f(U_i \times_U V) \longrightarrow f(U_i) \times_{f(U)} f(V)$$

is an isom.

Def A presheaf on a site T with values in a cat $\mathcal{C}$ is a contravariant functor $\mathrm{cat}(T) \to \mathcal{C}$.

morphism = natural transformation.

(Yoneda embedding: $\mathrm{cat}(T) \longrightarrow \mathrm{Pre}(T)$

$$X \longmapsto \underset{\substack{\shortparallel \\ h_X}}{\mathrm{Hom}_{\mathrm{cat}(T)}(-,X)}$$)

Def A presheaf $\mathcal{F} \in \mathrm{Pre}(T)$ is a sheaf if for every covering $\{U_i \to U\}$ in $\mathrm{cov}(T)$ we have an equalizer diagram in $\mathcal{C}$

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \underset{\mathcal{F}(\pi_1) = \pi_1^*}{\overset{\mathcal{F}(\pi_0) = \pi_0^*}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

section on U s = $\{$ collections of sections s_i on U_i s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

actually

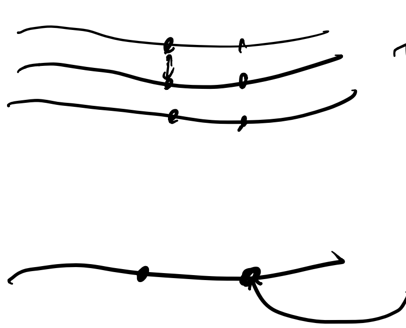
$$\mathcal{F}(\pi_1)(s_i) = \mathcal{F}(\pi_0)(s_j)$$

$$\begin{array}{ccc} & & s_i \\ U_i \times_U U_j & \xrightarrow{\pi_1} & U_i \\ & \searrow^{\pi_0} & U_j \end{array}$$

Étale topology

(small étale site)

Def An étale morphism (of schemes) is a morphs. $f: X \to Y$ which is flat, locally of finite presentation and unramified.



$\coprod \mathrm{Spec}(\bar{k})$ means, for $y \in Y$, $X_y = X \times_Y y$ is (the spectrum of) a sep. field ext. of $\kappa(y)$

product f.

$\mathrm{Spec}(\bar{k})$

$k[T]/T^p - 1$

<u>equiv</u> étale = lfp. + "formally étale"

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d] & \hat{X} \arrow[d] \\ I \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}\, R/I \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}\, R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

I nilpotent